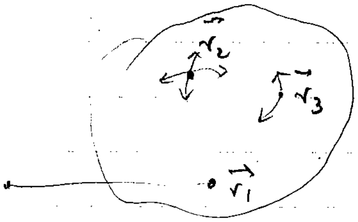
Rigid Bodies

? configuration
for a rigid body ?

Coordinates for a rigid body



body within which
relative distances
do not change

approximation to real bodies
for weak forces
and coarse time-resolution
space

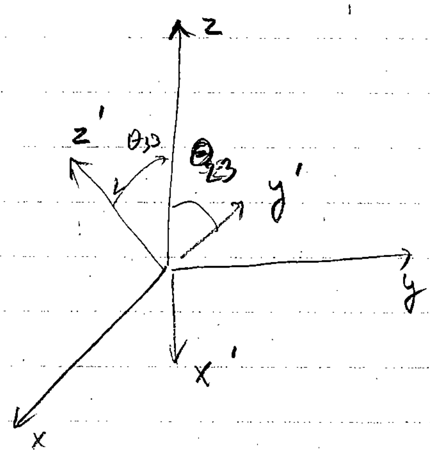
To position one chosen point, such as $\vec r_1$, 3 coordinates are needed. To position one more point after that, e.g. $\vec r_2$, 2 coordinates. The second point can move on a surface of a sphere of radius r_{21}. After that, the points of the body can still rotate on circles concentric with the $\vec r_{21}$ axis and we can restrain that by specifying the position of some point $\vec r_3$ with one coordinate.

After that, all points of the body have fixed positions.

Altogether, we need 6 coordinates to specify the configuration of a rigid body, 3 to specify the placement and 3 to specify orientation.

Constraints placed on a rigid body, may reduce the number of needed coordinates.

When describing the mass distibution within a rigid body it is convenient to use carthesian coordinates in a system associated with the body. Within that system the distibution never changes. However, the body coordinate axes will be generally oriented differently then for an external system. For the moment, we shall assume that the origins of the systems coincide.



′ coordinate system associated with the body

unprimed – external

unit vectors in the directions of the axes $\vec{i}, \vec{j}, \vec{k} \equiv \vec{u}_1, \vec{u}_2, \vec{u}_3$

coordinates $x, y, z \equiv x_1, x_2, x_3$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \equiv \sum_l x_l \vec{u}_l \equiv x_l \vec{u}_l$$

Unless otherwise specified or an explicit summation sign is used, the repeated indices imply a summation over those indices

Simultaneously, we have

$$\vec{r} = x'_k \vec{u}'_k$$

Now

$$\vec{r}\cdot\vec{u}_k = x_l \vec{u}_l\cdot\vec{u}_k = x_l \delta_{lk} = x_k$$

So $\vec{r} = (\vec{r}\cdot\vec{u}_l)\cdot\vec{u}_l$

In particular ↙ angle between primed & unprimed axes

$$\vec{u}'_k = (\vec{u}'_k\cdot\vec{u}_l)\cdot\vec{u}_l = \cos\theta_{kl}\,\vec{u}_l$$

Now we have

$$\vec{u}'_k \vec{u}'_j = \delta_{kj} = \cos\theta_{kl}\,\vec{u}_l \cos\theta_{ji}\,\vec{u}_i$$

$$= \cos\theta_{kl}\cos\theta_{ji}\,\delta_{li}$$

$$= \cos\theta_{kl}\cos\theta_{jl}$$

$k=j \Rightarrow 1 = \sum_l \cos^2\theta_{kl}$ 3 eqs

$k\neq j \Rightarrow 0 = \sum_l \cos\theta_{kl}\cos\theta_{jl}$ $\frac{2\times 3}{2} = 3$ eqs

altogether $3\times 3 = 9$ angles

Also

$$\vec{u}_l = (\vec{u}_l\cdot\vec{u}'_k)\vec{u}'_k = \cos\theta_{kl}\,\vec{u}'_k$$

$$\vec{u}_l\vec{u}_n = \delta_{ln} = \cos\theta_{kl}\,\vec{u}'_k \cos\theta_{mn}\,\vec{u}'_m = \cos\theta_{kl}\cos\theta_{mn}\,\delta_{km}$$

$$= \sum_k \cos\theta_{kl}\cos\theta_{kn}$$

Vector $\vec{r}$

$$x'_k = \vec{r}\cdot\vec{u}'_k = x_l \vec{u}_l \cdot \vec{u}'_k$$

$$= x_l \cos\theta_{kl}$$

Introducing a matrix $\bar{\bar{A}}$ with elements

$$A_{kl} = \cos\theta_{kl}$$

row index ↑ ↑ column index

$$\bar{\bar{A}} = \begin{pmatrix} \cos\theta_{11} & \cos\theta_{12} & \cos\theta_{13} \\ \cos\theta_{21} & \cos\theta_{22} & \cos\theta_{23} \\ \cos\theta_{31} & \cos\theta_{32} & \cos\theta_{33} \end{pmatrix}$$

$$x'_k = A_{kl} x_l$$

$k=1,2,3$ this represents a linear transformation in 3 dim ∴ $\bar{\bar{A}}$ is called a transformation mtx

We have

$$\sum_l A_{kl} A_{jl} = \sum_l \cos\theta_{kl} \cos\theta_{jl} = \delta_{jk}$$

$$\sum_l A_{kl} A_{jl} = \delta_{jk}$$

A transformation with such a property is termed orthogonal

Also

$$\sum_k A_{kl} A_{kn} = \delta_{ln}$$

Introducing a transposed matrix $\bar{\bar{A}}^T$, such that

$$(\bar{\bar{A}}^T)_{\ell j} = \bar{\bar{A}}_{j\ell}$$

we can write the orthogonality condition as

$$A_{kl} A^T_{lj} = \delta_{jk} \quad \text{or} \quad \bar{\bar{A}} \cdot \bar{\bar{A}}^T = \bar{\bar{1}}$$

also $\bar{\bar{A}}^T \cdot \bar{\bar{A}} = 1$

where $\bar{\bar{1}}$ is a unit matrix

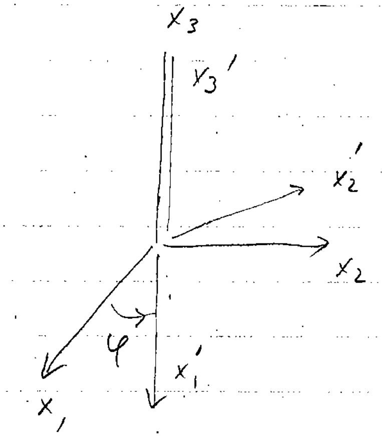
$$\bar{\bar{1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The orthogonality condition ensures that the scalar products remain the same no matter what the coordinate system is used

$$\vec{x} = x_k u_k \qquad \vec{y} = y_k u_k \qquad \vec{x} \cdot \vec{y} = x_k y'_k$$

$$\vec{x} \cdot \vec{y} = x'_k y''_k = A_{kl} x_l A_{kn} y_n = \delta_{ln} x_l y_n = x_n y_n$$

Example: Coordinate systems related by a rotation about the z-axis



$$x_1' = x_1 \cos\varphi + x_2 \sin\varphi$$

$$x_2' = -x_1 \sin\varphi + x_2 \cos\varphi$$

$$x_3' = x_3$$

$$A_{11} = \cos\varphi \qquad A_{12} = \sin\varphi \qquad A_{13} = 0$$

$$A_{21} = -\sin\varphi \qquad A_{22} = \cos\varphi \qquad A_{23} = 0$$

$$A_{31} = 0 \qquad A_{32} = 0 \qquad A_{33} = 1$$

$$\bar{\bar{A}} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Orthogonality $\quad \sum_l A_{kl} A_{nl} = \delta_{kl} \quad ?$

$k=n=1 \qquad \cos^2\varphi + \sin^2 + 0 = 1$

$k=n=2 \qquad \sin^2\varphi + \cos^2\varphi + 0 = 1$

$k=n=3 \qquad 0 + 0 + 1 = 1$

$k=1 \; n=2 \qquad -\cos\varphi \sin\varphi + \sin\varphi \cos\varphi + 0 = 0$

$k=1 \; n=3 \qquad 0 + 0 + 0 = 0$

Transformation matrix as an operator

When the two coordinate systems coincide the coordinates for $\vec{r}$ in the set $\mathcal{O}$ and $\mathcal{O}'$ are the same. After the body rotation the coordinates in $\mathcal{O}'$ stay the same while those in $\mathcal{O}$ change. Thus we may view the relation

$$\underbrace{x'_k}_{\text{before body rotation}} = A_{ke} \overbrace{x_e}^{\text{after rotation}}$$

as a rotation transformation. Using the orthogonality relation

$$A_{kn} x'_k = A_{kn} A_{ke} x_e = \delta_{ne} x_e = x_n$$

$$A^T_{nk} x'_k = x_n$$

$$\vec{r} = \bar{\bar{A}}^T \underbrace{(\bar{r})'}_{}$$

vector with coordinates such as in body frame

that vector gets rotated in the manner prescribed by $\bar{\bar{A}}^T$ to produce $\vec{r}$

Inverse Transformation

$$\begin{tikzcd} \theta \arrow[r, "A"] & \theta' \arrow[l, bend left, "A^{-1}"] \end{tikzcd}$$

$$x_i' = A_{ik} x_k$$

$$x_k = A^{-1}_{kj} x'_j \qquad x'_j = \delta_{jn}$$

$$x'_i = A_{ik} A^{-1}_{kj} x'_j \qquad \Rightarrow \quad A_{ik} A^{-1}_{kj} = \delta_{ij}$$

$$\bar{\bar{A}}\,\bar{\bar{A}}^{-1} = \bar{\bar{I}}$$

$$x_k = A^{-1}_{kj} A_{ji} x_i \qquad \Rightarrow \quad A^{-1}_{kj} A_{ji} = \delta_{ki}$$

$$\bar{\bar{A}}^{-1}\bar{\bar{A}} = \bar{\bar{I}}$$

We have $\qquad \bar{\bar{A}}^{-1}\bar{\bar{A}} = \bar{\bar{A}}^{T}\bar{\bar{A}}$

$$A_{jk} A_{ik} = \delta_{ji}$$

$$A_{jk} A^T_{ki} = \delta_{ji} \qquad \bar{\bar{A}}^T = \bar{\bar{A}}^{-1}$$

Inverse of an orthogonal matrix is equal to the transpose of the matrix.] equivalent def.

Other special matrices

$$\bar{\bar{A}}^T = \bar{\bar{A}} \quad \Leftrightarrow \quad A_{ij} = A_{ji} \qquad \text{symmetric}$$

$$\bar{\bar{A}}^T = -\bar{\bar{A}} \quad \Leftrightarrow \quad A_{ij} = -A_{ji} \qquad \text{antisymmetric}$$

note, diagonal elements of antisymmetric mtx $A_{ii} = 0 \qquad i = 1, 2, 3$

Superposition of transformations

$$\theta \to \theta' \to \theta''$$

$$x'_k = B_{kj}\, x_j$$

$$x''_i = A_{ik}\, x'_k = A_{ik}\, B_{kj}\, x_j$$

$$x''_i = C_{ij}\, x_j$$

$$C_{ij} = A_{ik}\, B_{kj}$$

the transformation matrix

$$\bar{\bar{C}} = \bar{\bar{A}} \cdot \bar{\bar{B}}$$

The multiplication is not commutative, i.e. generally $\bar{\bar{B}} \cdot \bar{\bar{A}} \neq \bar{\bar{A}}\,\bar{\bar{B}}$.

However, the multiplication is associative

$$\begin{tikzcd}
\theta \arrow[r, "C"] \arrow[rrr, bend right] & \theta' \arrow[r, "B"] & \theta'' \arrow[r, "A"] & \theta'''
\end{tikzcd}$$

$$\bar{\bar{D}} = \bar{\bar{A}} \cdot \bar{\bar{B}} \cdot \bar{\bar{C}} = \bar{\bar{A}}\,(\bar{\bar{B}} \cdot \bar{\bar{C}}) = (\bar{\bar{A}} \cdot \bar{\bar{B}}) \cdot \bar{\bar{C}}$$

$$D_{ij} = (A_{ik}\, B_{kl})\, C_{lj}$$

demonstrate here noncom

Transformation of linear operators

Let us consider a linear operator within one coordinate system, O,
Examples: multiplication by a constant, rotation, projection on an axis, etc.

$y_i = F_{ik} x_k$ $\qquad O$

How does the operator transform under the change of the coordinate system?

$y'_j = F'_{je} x'_e$ $\qquad O'$

Relation between F' and F?

$A_{jn} y_n = F'_{je} A_{em} x_m$

$A_{jn} F_{nk} x_k = F'_{je} A_{em} x_m$ $\qquad x_k = \delta_{ks}$

$A_{jn} F_{ns} = F'_{je} A_{es}$ $\qquad \times A^{-1}_{su}$

$A_{jn} F_{ns} A^{-1}_{su} = F'_{ju}$ $\qquad \Rightarrow \bar{\bar{F}}' = \bar{\bar{A}} F \bar{\bar{A}}^{-1}$

$\bar{\bar{I}}' = \bar{\bar{A}} \bar{\bar{I}} \bar{\bar{A}}^{-1} = \bar{\bar{I}}$

Determinants

$$|\bar{\bar{A}}|$$

$$|\bar{\bar{A}} \cdot \bar{\bar{B}}| = |\bar{\bar{A}}| \cdot |\bar{\bar{B}}|$$

$$|\bar{\bar{1}}| = 1 \qquad |\bar{\bar{A}}^T| = |\bar{\bar{A}}|$$

$$1 = |\bar{\bar{1}}| = |\bar{\bar{A}} \, \bar{\bar{A}}^{-1}| = |\bar{\bar{A}}| \, |\bar{\bar{A}}^{-1}|$$

$$|\bar{\bar{A}}| = |\bar{\bar{A}}^T| \underset{\substack{\uparrow \\ \text{for an orthogonal matrix}}}{=} |\bar{\bar{A}}^{-1}| = |\bar{\bar{A}}|^{-1}$$

$$1 = |\bar{\bar{A}}|^2 \Rightarrow |\bar{\bar{A}}| = \pm 1$$

Only transformations with $| \; | = +1$ can represent rotations

A rotation may be represented as a superpositions of tiny rotations

$$\bar{\bar{A}} = \underset{\uparrow}{A_1} \cdots \underset{\uparrow}{A_n}$$

tiny rotations, nearly $\bar{\bar{1}}$
must have $|\bar{\bar{A}}_i| = 1$

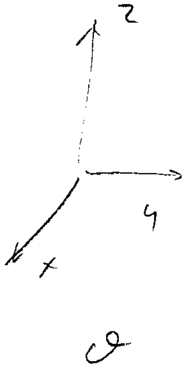
$$\Rightarrow |\bar{\bar{A}}| = 1$$

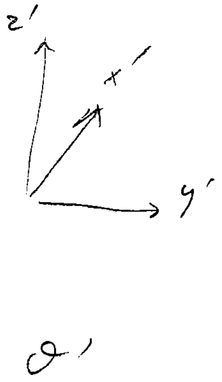
Transformations with $| \; | = -1$ $\Rightarrow$ rotations followed by a reflection Eq

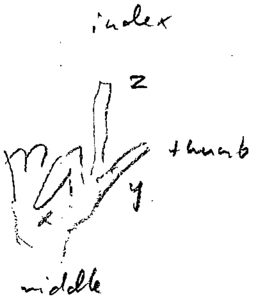
↓ change handedness of a coordinate system
~~cannot be accomplished~~

$$\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

$x' = -x$
$y' = y$
$z' = z$



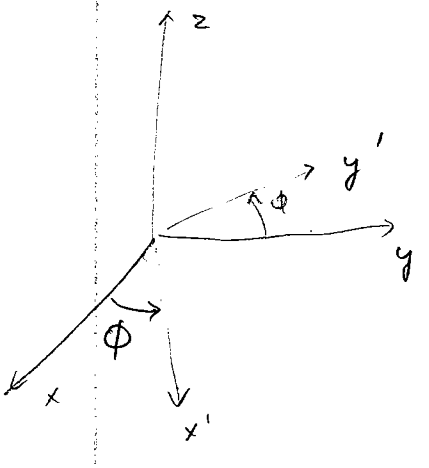


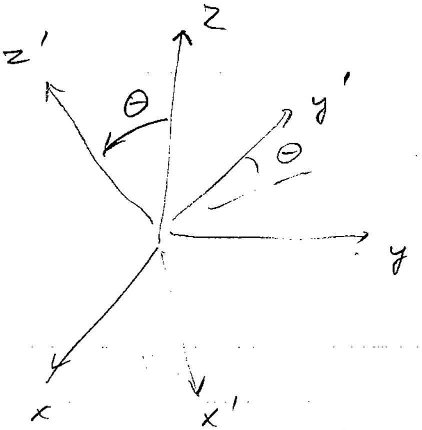


Right-handed system turned into left-handed

Euler Angles
3 parameters sufficient to specify a general rotation transformation from 𝒪 to 𝒪'

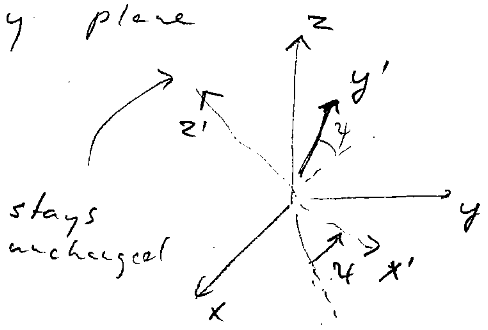
One choice: Euler angles





F stays unchanged

But still x' stays with xy plane
Can't be most general
Final rotation
by ψ around z'



Full transformation matrix.

$$\bar{\bar{A}} = \bar{\bar{B}}\;\bar{\bar{C}}\;\bar{\bar{D}}$$

$\uparrow \nearrow \nearrow$

3 rotations

$$\bar{\bar{D}} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\bar{\bar{C}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$\bar{\bar{D}} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Inverse transformation matrix

$$\bar{\bar{A}}^{-1} = \bar{\bar{D}}^{-1}\bar{\bar{C}}^{-1}\bar{\bar{B}}^{-1} = \bar{\bar{D}}^{T}\bar{\bar{C}}^{T}\bar{\bar{B}}^{T}$$

Euler's theorem: The general displacement of a rigid body with one point fixed is a rotation about some axis.

$\mathcal{O}$ external
$\mathcal{O}'$ body-fixed

With regard to coordinate systems $\mathcal{O}$ & $\mathcal{O}'$ centered at the fixed point – any transf. from $\mathcal{O}$ to $\mathcal{O}'$ just a single rotation

change from one coordinate set to another with the same origin can be represented as a rotation about an axis ⟺ any vector along that axis would have the same components in the two systems

e.g. z axis
y axis
x axis
or any other

– axis
– angle of rotation

With $\vec{r} = (x_1, x_2, x_3)$

and $\vec{r}\,' = (x_1', x_2', x_3')$

For any $\bar{\bar{A}}$ a vector $\vec{r}$ must exist such that

$$\vec{r}\,' = \overset{\rightrightarrows}{A}\vec{r} = \vec{r}$$

I. e. any transformation matrix must have an eigenvector with

$$\overset{\rightrightarrows}{A}\vec{r} = \lambda\vec{r}$$

with an eigenvalue 1, $\lambda = 1$.

I.e.

$$(\bar{\bar{A}} - \lambda\bar{\bar{I}})\vec{r} = 0 \iff |\bar{\bar{A}} - \lambda\bar{\bar{I}}| = 0$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{23} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

A 3-order polynomial has 3 solutions $\lambda_1, \lambda_2, \lambda_3$, corresponding to three eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$
orthogonal

$$\bar{\bar{A}}\, \vec{v}_k = \lambda_k \vec{v}_k$$

i'th component of the k'th vector

$$A_{in} v_{nk} = \lambda_k v_{ik}$$

← no summation over k

↑ k'th vector

Constructing a matrix $\bar{\bar{V}}$

$$V_{nk} = v_{nk}$$

we can represent the eigenvalue eqs as

$$A_{in} V_{nk} = V_{ij} \delta_{jk} \lambda_k$$

and on applying $\bar{\bar{V}}^{-1}$ we can accomplish a diagonalization of $\bar{\bar{A}}$

$$V^{-1}_{mi} A_{in} V_{nk} = V^{-1}_{mi} V_{ij} \delta_{jk} \lambda_k = \delta_{mk} \lambda_k$$

$$\bar{\bar{V}}^{-1} \bar{\bar{A}}\, \bar{\bar{V}} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$\lambda_1 \lambda_2 \lambda_3 = |\bar{\bar{V}}|^{-1} |\bar{\bar{A}}| |\bar{\bar{V}}| = |\bar{\bar{A}}| = 1$$

3 eigenvectors, As the transformation preserves distances, then a unit vector should remain a unit vector after the transformation, so $\lambda_k = 1$??
But then $\bar{\bar{A}} \equiv \bar{\bar{I}}$ Any t. matrix should identical to $\bar{\bar{I}}$
Where's the mistake in reasoning?? λ's & v's might be complex! So we might have
$\lambda_1 \lambda_2 \lambda_3 = 1$ out of 3 complex eigenvalues
corresponding to 3 complex vectors a none in real space
only rotations by 90° can be represented with such reflections

cases
$\lambda = 1$
$\lambda = 1$ $\lambda = -1 \times 2$

or exactly a
just
rotations,
2 reflections

This is the situation in 2D. ??

$$(\bar{\bar{A}} - \bar{\bar{I}})\,\bar{\bar{A}}^T = \bar{\bar{I}} - \bar{\bar{A}}^T$$

$$|\bar{\bar{A}} - \bar{\bar{I}}|\;|\bar{\bar{A}}^T| = |\bar{\bar{I}} - \bar{\bar{A}}^T|$$

$$|\bar{\bar{A}} - \bar{\bar{I}}| = |\bar{\bar{I}} - \bar{\bar{A}}|$$

Generally: $|-\bar{\bar{B}}| = (-1)^n\,|\bar{\bar{B}}|$ where n is the dimension of the space

In 3D $|\bar{\bar{I}} - \bar{\bar{A}}| = (-1)^3\,|\bar{\bar{A}} - \bar{\bar{I}}|$

With $|\bar{\bar{A}} - \bar{\bar{I}}| = -|\bar{\bar{I}} - \bar{\bar{A}}| \Rightarrow |\bar{\bar{A}} - \bar{\bar{I}}| = 0$

$$0 = |\quad| = (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1)$$

Thus, there must be a unit eigenvalue & the Euler theorem is proven.

If we take $\lambda_3 = 1$ what about λ_1 & λ_2?

For $\bar{\bar{A}}$ real, if λ is a solution of $|\bar{\bar{A}} - \lambda\bar{\bar{I}}| = 0$, then also λ^* is, so $\lambda_2 = \lambda_1^*$

We also have $\lambda_1\lambda_2 = 1 \Rightarrow \lambda_1\lambda_1^* = 1$

$\Rightarrow |\lambda_1|^2 = 1 \Rightarrow \lambda_1 = e^{i\Phi}$, $\lambda_2 = e^{-i\Phi}$

$$\mathrm{Tr}\,\bar{\bar{A}} = \lambda_1 + \lambda_2 + \lambda_3 = 2\cos\Phi + 1$$

What is Φ?

In the vector basis, where the 3^{rd} points along the invariant axis, the mtx

$$\overline{\overline{A}}' = \begin{pmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2 angles need
to orient $\vec{n}$
+ angle of
rotation
around

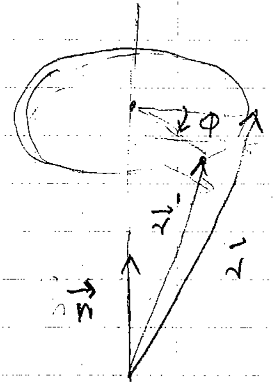
$$\mathrm{Tr}\,\overline{\overline{A}}' = 2\cos\Phi + 1$$

← now

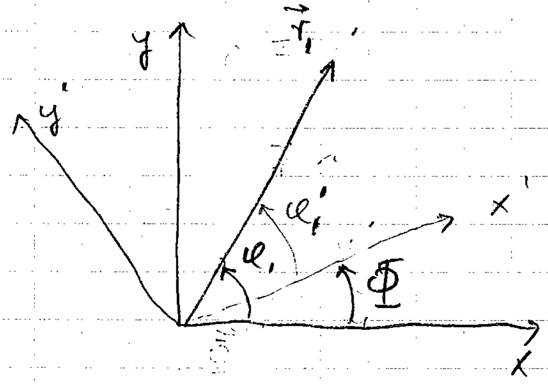
⇒ Φ is angle of rotation about the axis

↳ the angle, up to the sign, can be obtained from the trace ...

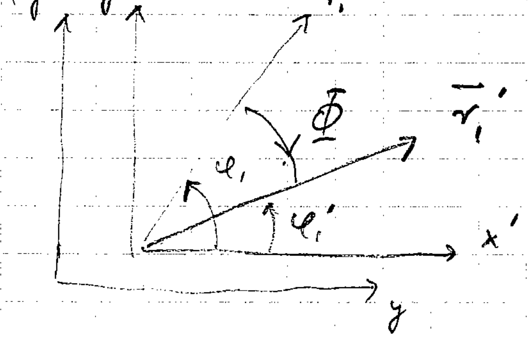
coordinate transf in terms of $\vec{n}$ and Φ



When a coordinate system is rotated one way, then the vector, relative to coordinates is rotated the other way.



In the frame of the axes the coordinate vectors



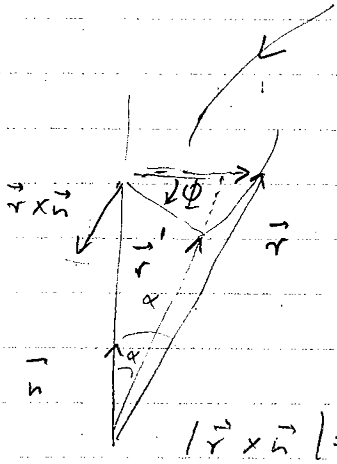
Coordinate transformation in terms of the axis and angle

$\vec{r}\,' = (x_1', x_2', x_3')$

$\vec{r} = (x_1, x_2, x_3)$

Three orthogonal axes in terms of $\vec{n}$ & $\vec{r}$

$\vec{n}$, $\vec{r} - (\vec{n}\cdot\vec{r})\vec{n}$ $\quad \vec{r}\times\vec{n}$



$$\vec{r}\,' = \vec{n}(\vec{n}\cdot\vec{r}) + (\vec{r} - (\vec{n}\cdot\vec{r})\vec{n})\cos\Phi$$
$$+ (\vec{r}\times\vec{n})\sin\Phi$$

$|\vec{r}\times\vec{n}| = r\sin\alpha$

$|\vec{r} - (\vec{n}\cdot\vec{r})\vec{n}| = r\sin\alpha$

$$\vec{r}\,' = \vec{r}\cos\Phi + \vec{n}(\vec{n}\cdot\vec{r})(1-\cos\phi) + (\vec{r}\times\vec{n})\sin\phi$$

Rotation formula

? Rotation angle in terms of Euler formulas

Trace of a matrix does not depend on the basis for a matrix

$$\mathrm{Tr}\,\bar{\bar{A}}' = \mathrm{Tr}\,\bar{\bar{A}} = \mathrm{Tr}(\bar{\bar{B}}\,\bar{\bar{C}}\,\bar{\bar{D}})$$

$\|$

in terms of Euler angles

$$\hookrightarrow \cos\frac{\Phi}{2} = \cos\frac{\phi+\psi}{2}\cos\frac{\theta}{2}$$

Rotations from Superpositions of Infinitesimal Transformations

depends on the axis of rotation

$$x_k' \approx x_k + \underbrace{\overset{\swarrow}{\epsilon_{ki}}\, x_i}_{\text{small addition}} = (\delta_{ki} + \epsilon_{ki})\, x_i$$

In vector form

$$\vec{x}\,' = (\bar{\bar{I}} + \bar{\bar{\epsilon}})\,\vec{x}$$

Two infintesimal rotations: $I + \epsilon_1$ & $\bar{\bar{I}} + \bar{\bar{\epsilon}}_2$

$$(\bar{\bar{I}} + \bar{\bar{\epsilon}}_1)(\bar{\bar{I}} + \bar{\bar{\epsilon}}_2) = \bar{\bar{I}}^2 + \bar{\bar{I}}\epsilon_2 + \bar{\bar{I}}\bar{\bar{\epsilon}}_1 + \bar{\bar{\epsilon}}_1\bar{\bar{\epsilon}}_2$$

$$= \bar{\bar{I}} + \bar{\bar{\epsilon}}_1 + \bar{\bar{\epsilon}}_2 = (\bar{\bar{I}} + \bar{\bar{\epsilon}}_2)(\bar{\bar{I}} + \bar{\bar{\epsilon}}_1)$$

↑ to the lowest order

Infintesimal rotations commute, in contrast to those for finite angle

In general $\bar{\bar{B}}\bar{\bar{C}} \neq \bar{\bar{C}}\bar{\bar{B}}$

For a rotation over 3 infintesimal Euler angles

$$\bar{\bar{A}} = \begin{pmatrix} 1 & d\psi & 0 \\ -d\psi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & d\theta \\ 0 & -d\theta & 1 \end{pmatrix} \begin{pmatrix} 1 & d\varphi & 0 \\ -d\varphi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & d\varphi + d\psi & 0 \\ -d\varphi - d\psi & 1 & d\theta \\ 0 & -d\theta & 1 \end{pmatrix} \Rightarrow \bar{\bar{\epsilon}} = \begin{pmatrix} 0 & d\varphi + d\psi & 0 \\ -d\varphi - d\psi & 0 & d\theta \\ 0 & -d\theta & 0 \end{pmatrix}$$

If $\bar{\bar{A}} = \bar{\bar{I}} + \bar{\bar{\epsilon}}$, then $\bar{\bar{A}}^{-1} = \bar{\bar{I}} - \bar{\bar{\epsilon}}$, because

$$\bar{\bar{A}}\bar{\bar{A}}^{-1} = (\bar{\bar{I}} + \bar{\bar{\epsilon}})(\bar{\bar{I}} - \bar{\bar{\epsilon}}) = \bar{I} - \cancel{\bar{\bar{\epsilon}}} + \cancel{\bar{\bar{\epsilon}}} - \bar{\bar{\epsilon}}^2 = \bar{\bar{I}}$$

Since

$$\bar{\bar{A}}^{-1} = \bar{\bar{A}}^{T} \iff \bar{\bar{I}} - \bar{\bar{\epsilon}} = 1 + \bar{\bar{\epsilon}}^{T}$$

we must have

$$\bar{\bar{\epsilon}}^{T} = -\bar{\bar{\epsilon}},$$

i.e. the mtx $\bar{\bar{\epsilon}}$ is antisymmetric
Without the loss of generality, we may represent it as

$$\bar{\bar{\epsilon}} = \begin{pmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{pmatrix}$$

$$\vec{r}\,' - \vec{r} \equiv d\vec{r} = \bar{\bar{\epsilon}}\,\vec{r}$$

$$dx_1 = x_2\, d\Omega_3 - x_3\, d\Omega_2$$

$$dx_2 = x_3\, d\Omega_1 - x_1\, d\Omega_3$$

$$dx_3 = x_1\, d\Omega_2 - x_2\, d\Omega_1$$

$$d\vec{r} = \vec{r} \times d\vec{\Omega}$$

What is $d\vec{\Omega}$?

Rotation formule for small $d\phi$

$\cos d\phi \approx 1$, $\sin d\phi \approx d\phi$

$$\vec{r}\,' = \vec{r} + (\vec{r}\times\vec{n})\,d\phi$$

$$d\vec{\Omega} = \vec{n}\,d\phi$$

$$d\vec{r} = \vec{r}\times d\vec{\Omega} = -d\vec{\Omega}\,\underline{\times\,\vec{r}}$$

$$\bar{\bar{\epsilon}} = -d\Omega_k \bar{\bar{M}}_k = -d\phi\cdot n_k \bar{\bar{M}}_k$$

$$\bar{\bar{M}}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \bar{\bar{M}}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\bar{\bar{M}}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Infinitesimal rotation generators

$d\Omega_k M_{ijk} r_j$

$\parallel$

If we write

$$(d\vec{\Omega}\times\vec{r})_i = \varepsilon_{ikj}\, d\Omega_k\, r_j = -d\Omega_k\, \varepsilon_{ijk}\, r_j$$

where ε_{ikj} is the permutation symbol, equal to to 0 when any 2 indices are equal and otherwise either +1 or -1, if ikj is an even or odd permutation of 1, 2, 3, then

$$M_{ijk} = -\varepsilon_{ijk} \qquad \bar{\epsilon} = -d\phi\,\bar{n}\,\bar{\bar{M}}$$

↑

k'th mtx 321 ok 3rd rank tensor

Changes in the vector during rotation

$$\epsilon_{ij} \doteq + d\phi\, n_k\, \varepsilon_{ijk}$$

$$d r_i = \epsilon_{ij}\, r_j = d\phi\, n_k\, \varepsilon_{ijk}\, r_j$$

try to jump this

$$\frac{d r_i}{d\phi} = \left(\varepsilon_{ijk}\, n_k\right) r_j = N_{ij}\, r_j$$

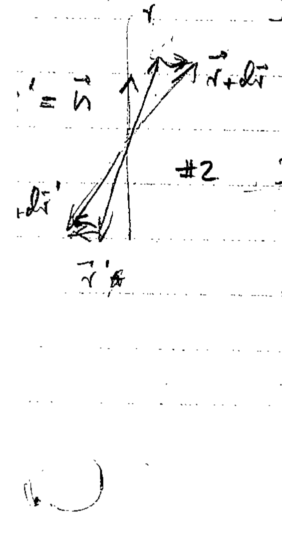
$$\frac{d\vec{r}}{d\phi} = \bar{\bar{N}}\,\vec{r} \qquad \bar{\bar{N}} = -n_k \bar{\bar{M}}_k = -\vec{n}\,\bar{\bar{M}}$$

$\uparrow$ axis can change during rotation

Note:

\#1 Above transformations refer to a case when the coordinate system is rotated by an angle $d\phi$. The vector, as perceived in the coordinate system, rotates by $-d\phi$. If an active rotation of the vector is done then the above results hold with $d\phi \to -d\phi$. In particular

$$d\vec{r} = d\vec{\Omega} \times \vec{r}$$



\#2 If improper transformations, $|\bar{\bar{A}}| = -1$, are allowed then $d\Omega$ transforms differently than vectors. Thus under an inversion

$$\bar{\bar{S}} = -\bar{\bar{I}} = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix}$$

$r_k' = A_{ki}\, r_i$

$d\Omega_k' = |\bar{\bar{A}}|\, A_{ki}\, d\Omega_i$

pseudovector

$$r_k' = S_{ki}\, r_i = -r_k \qquad d r_k' = S_{ki}\, d r_i = -d r_k$$

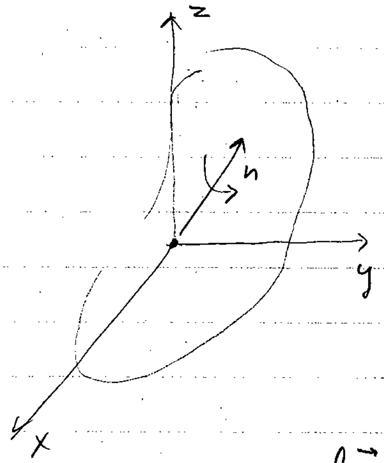
$$\Rightarrow d\Omega'_j = d\Omega_j \qquad d r_k = \varepsilon_{klj}\, d\Omega_j\, r_l \qquad -d r_k = \varepsilon_{klj}\, d\Omega'_j\, (-r_l)$$

#3 The infintesimal rotation operators satisfy commutation rules

$$\bar{\bar{M}}_i \bar{\bar{M}}_j - \bar{\bar{M}}_j \bar{\bar{M}}_i \equiv [\bar{\bar{M}}_i, \bar{\bar{M}}_j] = \varepsilon_{ijk} \bar{\bar{M}}_k$$

$[\cdot,\cdot]$ - commutator $\quad [\bar{\bar{A}}, \bar{\bar{B}}] \equiv \bar{\bar{A}}\bar{\bar{B}} - \bar{\bar{B}}\bar{\bar{A}}$

Rate of change as a body rotates



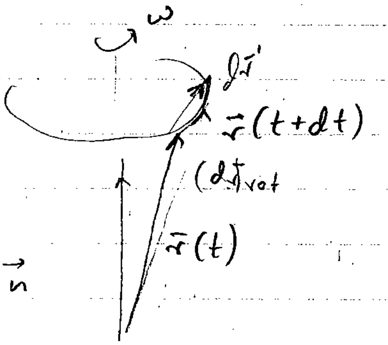
$\vec{r}\,' = (x_1' \; x_2' \; x_3')$ body fixed frame

$\vec{r} = (x_1 \; x_2 \; x_3)$ external frame

body rotates about a common center for the coordinates

$$d\vec{r} = d\vec{r}\,' + (d\vec{r})_{rot}$$

↑ there even when $d\vec{r}\,' = 0$



$$(d\vec{r})_{rot} = d\vec{\Omega} \times \vec{r}$$

$$d\vec{r} = d\vec{r}\,' + d\vec{\Omega} \times \vec{r}$$

$$\left(\frac{d\vec{r}}{dt}\right|_{0} = \left(\frac{d\vec{r}}{dt}\right|_{0'} + \frac{d\vec{\Omega}}{dt} \times \vec{r}$$

$$\vec{v} = \vec{v}\,' + \vec{\omega} \times \vec{r}$$

This will be valid for any vector compared in the 2 frames, not just position

$$\left(d\vec{G}\right)_{\vartheta} = \left(d\vec{G}\right)_{\vartheta'} + \left(d\vec{G}\right)_{rot} = \left(d\vec{G}\right)_{\vartheta'} + d\vec{\Omega}\times\vec{G}$$

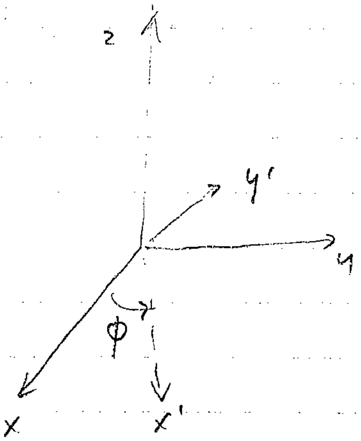
$$\left(\frac{d\vec{G}}{dt}\right)_{\vartheta} = \left(\frac{d\vec{G}}{dt}\right)_{\vartheta'} + \underbrace{\vec{\omega}\times\vec{G}}$$

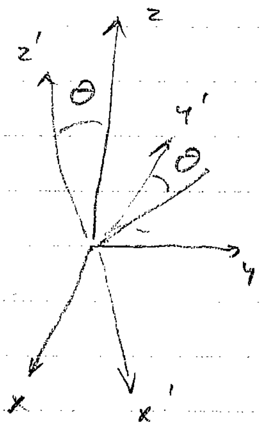
depending on application can be calculated in body or external frame

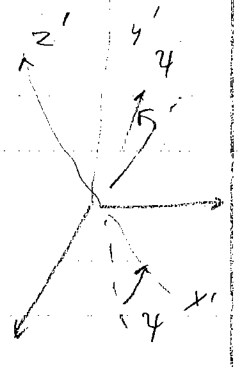
$$\left(\frac{d\vec{G}}{dt}\right)_{\vartheta} = \left(\frac{\partial\vec{G}}{dt}\right)_{\vartheta'} + \vec{\omega}\times\vec{G}$$

Schematically

$$\frac{d}{dt} = \frac{d}{dt}' + \vec{\omega}\times$$







Carthesian $\vec{\omega}'$ components in terms of Euler angular vel.

If only $\dot{\psi}$ finite

$\omega_z' = \dot{\psi}$ $\omega_x' = \omega_y' = 0$

If only $\dot{\phi}$ finite $\omega_z = \dot{\phi}$ $\omega_x = \omega_y = 0$

$\omega_z' = \dot{\phi}\cos\theta$ $\omega_y' = \dot{\phi}\sin\theta\cos\psi$ $\omega_x' = \dot{\phi}\sin\theta\sin\psi$

If only $\dot{\Theta}$ finite

$\omega_z' = 0$ $\omega_x' = \dot{\Theta}\cos\psi$ $\omega_y' = -\dot{\Theta}\sin\psi$

Overall, if all angles change

$$\omega_x' = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi$$

$$\omega_y' = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi$$

$$\omega_z' = \dot\phi \cos\theta + \dot\psi$$

Centrifugal and Coriolis Forces

$$\underset{\uparrow\ \text{in a system rotating at the frequency } \omega}{\vec v = \vec v'} + \vec\omega \times \vec r \qquad \vec\omega \text{ fixed vector in either frame}$$

$$\vec a = \frac{d\vec v}{dt} = \frac{d}{dt}\left(\vec v' + \vec\omega' \times \vec r'\right)_{\vartheta'} + \vec\omega \times (\vec v' + \vec\omega \times \vec r)$$

$$= \vec a' + \vec\omega' \times \vec v' + \vec\omega \times \vec v' + \vec\omega \times (\vec\omega \times \vec r)$$

$$= \vec a' + 2\vec\omega \times \vec v' + \vec\omega \times (\vec\omega \times \vec r)$$

$$\vec F = m\vec a = m\vec a' + 2m\,\vec\omega \times \vec v' + m\vec\omega \times (\vec\omega \times \vec r)$$

$$m\vec a' = \underset{\substack{\uparrow \\ \text{effective force} \\ \text{in a rotating} \\ \text{system}}}{\vec F_{eff}} = \vec F \overset{\vec F_{cf}}{\underbrace{- m\vec\omega \times (\vec\omega \times \vec r)}_{\text{centrifugal force}}} \overset{\vec F_{cor.}}{\underbrace{- 2m(\vec\omega \times \vec v')}_{\text{coriolis force}}}$$

The Coriolis force is present if the particle moves in the rotating frame.
The centrifugal force is present irrespective of a motion

$$\vec{F}_{cf} = -m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

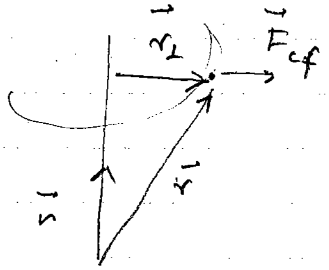
follows from the identity
for antisym symbol in homework.

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

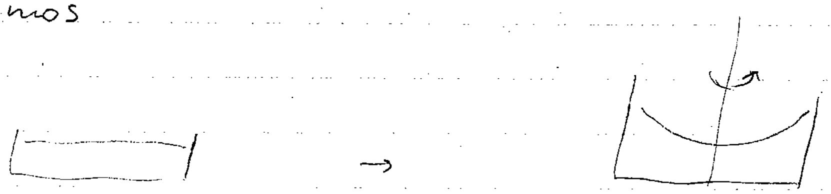
$$\vec{F}_{cf} = -m\left[(\vec{\omega}\cdot\vec{r})\vec{\omega} - \omega^2\vec{r}\right]$$

$$= m\omega^2\left[\vec{r} - \vec{n}\,(\vec{n}\cdot\vec{r})\right] = m\omega^2\,\vec{r}_\perp$$

$$\vec{F}_{cf} \perp \vec{\omega}, \vec{n}$$



demos ←



The effective force needs to be ⊥ to the surface of the water in the rotating frame. Otherwise the water can accelerate along the surface to the change in the surface shape.

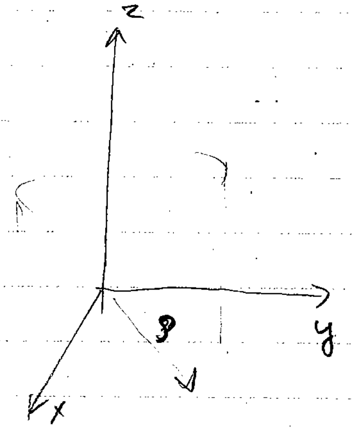
The surface of the water will align with the equipotential surface for the effective force in the rotating system, if such surface exists

$$\vec{F}_{ef} = m\underset{\substack{\uparrow\\ \text{small amount of water}}}{\vec{g}} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) = m\vec{g}_{ef}$$

$$\vec{g}_{ef} = \underline{\vec{g} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})}$$ effective gravitational acceleration

surface of the water needs to be locally ⊥ to $\vec{g}_{ef}$.

Exercise: Determine the shape for the surface of water in a cylindrical rotating pail.



$$\vec{F}_{ef} = -mg\,\vec{u}_z + m\omega^2 \rho\,\vec{u}_\rho$$

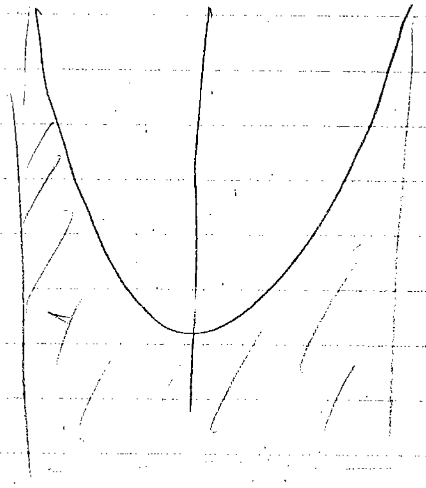
$$\vec{F}_{ef} = -\nabla V_{ef}$$

$$V_{ef} = mgz - \frac{1}{2} m\omega^2 \rho^2 + V_0$$

surface of constant V:

$$V_{ef} = mgz - \frac{1}{2} m\omega^2 \rho^2 + V_0 = const$$

$$z = \frac{\omega^2}{2g}\rho^2 + z_0$$ eq. of paraboloid



Newton observing the surface of water in a rotating pails came up with the concepts of inertial and noninertial frames of reference

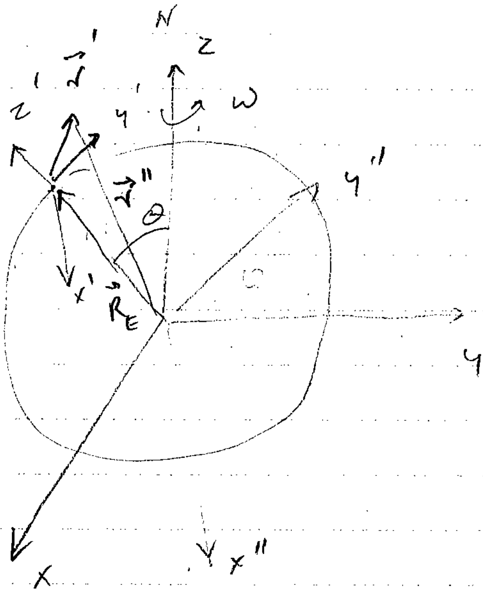
Sun moves towards West

Earth rotates towards East

O & O'' origin at Earth's center
O' at surface

Motion on the Earth

y' points towards the N pole



θ - colatitude

Note: axes not quite aligned!

angle

actual # of rotations (in the apparent one eaten up by the rotation around the Sun)

$$\omega = \frac{2\pi}{24\times3600} \times \frac{366.25}{365.25}$$

apparent # of rotations per year

duration of a year in sec

$$= 7.292 \times 10^{-5}\,s^{-1}$$

$$\frac{\vec F_{ef}}{m} = \vec g_{ef} = \vec g + \omega^2 \vec r_\perp \simeq \vec g + \omega^2 \vec R_{E\perp}$$

$$\Delta g = \omega^2 R_\perp = \omega^2 R_E \sin\theta$$

↑ 0.3% g

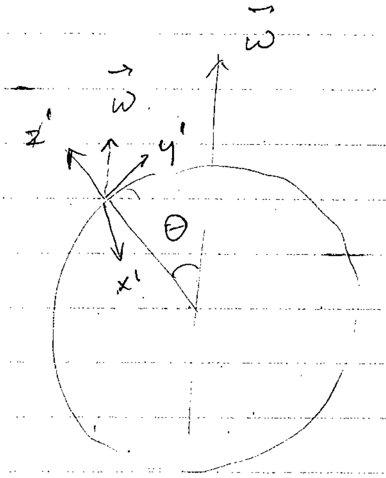
$$\omega^2 R_E = 5\times10^{-9}\,s^{-2} \times 6\times10^6\,m = 0.03\,\frac{m}{s^2}$$

$$R_E = \frac{2\pi R_E}{2\pi} = \frac{4\times10^7\,m}{2\pi} = 6.4\times10^6\,m$$

actually $R_E = 6{,}371$ km

Small, causes a finite buldging of Earth surface aligning with the equipotential surface of V_{ef}

The Coriolis force



$$\omega_z' = \omega \cos\theta$$
$$\omega_y' = \omega \sin\theta$$
$$\omega_x' = 0$$

$$\vec{F}_{cor} = -2m\,\vec{\omega}\times\vec{v}\,' = 2m\,\vec{v}\,'\times\vec{\omega}$$

$$\omega \begin{vmatrix} u_x' & u_y' & u_z' \\ v_x' & v_y' & v_z' \\ 0 & \sin\theta & \cos\theta \end{vmatrix}$$

$$\vec{F}_{cor} = 2m\omega\Big[(v_y'\cos\theta - v_z'\sin\theta)\vec{u}_x' - v_x'\cos\theta\,\vec{u}_y' + v_x'\sin\theta\,\vec{u}_z'\Big]$$

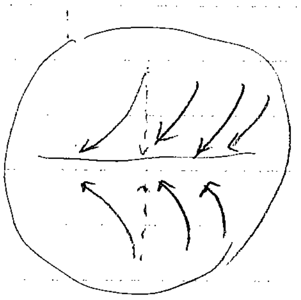
N hemisphere
$\theta < \frac{\pi}{2}$

Velocity dir		Force
N	$v_y' > 0$	East
E	$v_x' > 0$	South + up
S	$v_y' < 0$	West
W	$v_x' < 0$	North + Down
U	$v_z' > 0$	West
D	$v_z' < 0$	East

For motion ∥ to earth's surface, the ptcle is always deflected to the right in N hemisphere and to the left in the S hemisphere, I.e. the direction is to oppose a rotation with the frame.

Wind patterns are striking examples of the effects of the coriolis force

Global pattern. Heated air at the equator rises up, Colder Air from northern regions comes in



prevailing wind direction SW

Smaller scale patterns

N

Typhoon
hurricane