Epicycles… the short form.
For lurid details, see [CO pp.1018-1030]

Define an effective potential:
\[ \Phi_{\text{eff}}(R, z) \equiv \Phi(R, z) + \frac{J_z^2}{2R^2} \]

\[ \ddot{R} = -\frac{\partial \Phi_{\text{eff}}}{\partial R} \]

\[ \ddot{z} = -\frac{\partial \Phi_{\text{eff}}}{\partial z} \]

Taylor series expansion around position of minimum \( \Phi_{\text{eff}} \) (circular orbit):
\[ \Phi_{\text{eff}}(R, z) \approx \Phi_{\text{eff}, m} + \frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2} \bigg|_m \rho^2 + \frac{1}{2} \frac{\partial^2 \Phi_{\text{eff}}}{\partial z^2} \bigg|_m z^2 + \cdots \]

\[ \Phi_{\text{eff}}(R, z) \approx \Phi_{\text{eff}, m} + \frac{1}{2} \kappa^2 \rho^2 + \frac{1}{2} \nu^2 z^2. \]

\[ \rho \approx -\kappa^2 \rho \]

\[ \ddot{z} \approx -\nu^2 z \]

Conservation of specific angular momentum \( J_Z = R^2 \, d\phi/dt \)

Circular symmetry \( \implies \) independent of \( \phi \)

\[ \frac{d^2 \mathbf{r}}{dt^2} = -\nabla U(R, z) = -\frac{\partial U}{\partial R} \hat{e}_R - \frac{1}{R} \frac{\partial U}{\partial \phi} \hat{e}_\phi - \frac{\partial U}{\partial z} \hat{e}_z \]

Separate \( d^2 \mathbf{r}/dt^2 \) into \( R, \phi, z \) components \( \implies \) 3 equations.

\[ \frac{d^2 \mathbf{r}}{dt^2} = \left( \ddot{R} - R \dot{\phi}^2 \right) \hat{e}_R + \frac{1}{R} \frac{\partial \left( R^2 \dot{\phi} \right)}{\partial t} \hat{e}_\phi + \ddot{z} \hat{e}_z \]
Harmonic oscillation in \( R, \phi, z \) about circular orbit (Epicycles)

\[
\ddot{\rho} \simeq -\kappa^2 \rho \\
\ddot{z} \simeq -\nu^2 z
\]

\[
\rho(t) = R(t) - R_m = A_R \sin \kappa t \\
z(t) = A_z \sin (\nu t + \zeta)
\]

\[
\phi(t) = \phi_0 + \frac{J_z}{R_m^2} t + \frac{2J_z}{\kappa R_m^3} A_R \cos \kappa t = \phi_0 + \Omega t + \frac{2\Omega}{\kappa R_m} A_R \cos \kappa t
\]

\[
R_m = R \text{ at min. } \Phi_{\text{eff}} \\
\Omega = \text{circular ang. vel.}
\]

In inertial frame:

Orbits closed if:

\[
m \left( \Omega - \Omega_{lp} \right) = n \kappa
\]

\[
\Omega_{lp}(R) = \Omega(R) - \frac{n}{m} \kappa(R)
\]

Viewed from frame rotating with \( \Omega_{lp} \):

Two ways to line up closed elliptical orbits (as seen from frame rotating with \( \Omega_{lp} \))
Basic nature of a density wave


- At each $R_m$, stars’ positions in epicycles are forced into a specific pattern by gravitational potential of spiral arm.
- Sum of positions of stars at this $R_m$ forms an ellipse rotating at pattern speed.
- Spiral density pattern is sum of many ellipses, all rotating at same pattern speed.

**Figure 2** Slow $m = 2$ kinematic wave on a ring of test particles, all revolving clockwise (like the 12 shown) with mean angular speed $\Omega$ in strictly similar and nearly circular orbits. The small elliptical “epicycles,” traversed counterclockwise in the above sequence of snapshots separated in time by exactly one-quarter of the period $2\pi/\kappa$ of radial travel along each orbit, depict the apparent motions of these particles relative to their mean orbital positions or “guiding centers.” Drawn for the case $\kappa = \sqrt{2}\Omega$—or one where the rotation speed $V(r) = r\Omega(r) = \text{const}$ at neighboring radii—the diagram emphasizes that the oval locus of such independent orbiters advances in longitude considerably more slowly than the particles themselves. That precession rate equals $\Omega - \kappa/2$, as one can verify at once by comparing the last frame with the first.
Lin & Shu’s theory

Perturbed form of collisionless Boltzmann equation. “quite complicated”

Fluid hydrodynamics

Resultant gravitational field due to stars and gas

Total material needed to maintain the resultant gravitational field

Density response of the stellar disk

Density response of the gaseous disk

Total response in the distribution of the matter in the disk

This equation serves to specify the wave properties completely

Figure 12.26. The calculational scheme used to calculate the normal modes of oscillation of a disk galaxy. (After C. C. Lin.)

Figure 12.27. Contours of equal density excesses above the average value around a circle in a typical spiral mode. The dashed circle gives the radius where the material rotates at the same speed as the wave pattern.
Density wave theory interprets most spirals as 2-armed

- 4-armed pattern is \( n / m = 1 / 4 \)
- exists over a narrow range of radius.
  \( \Rightarrow \) less likely to be seen.

- Actual 4-armed spirals are superposition of two 2-armed patterns
Spiral Structure

[CO 23.3]

Grand design (10%)
M51

Multi-arm (60%)
M101

Flocculent (30%)
NGC 2841

Inner rings
NGC 7096

M81

Outer Ring
NGC 4340
M81 spiral structure at different wavelengths

UV: hot stars

Visible: stars + obscuration

Near IR: late-type stars

Old red population shows small but real spiral density enhancement.

21 cm: HI
Passage of gas through spiral arms

Figure 12.33. The streamlines of gas from a theoretical model of the spiral galaxy M81 (NGC3031, type Sb). (From H. Visser, *Astr. Ap.*, 88, 1980, 159.)

Calculated streamlines for gas
Response of gas to density waves

- Simple pendulum model
  - Each pendulum = 1 gas cloud
  - For large amplitude forcing, pendulums collide.
    - shock fronts in spiral arms

- HI map (right) shows velocity jumps at spiral arms.
Orbits in Barred Spirals

• Gas avoids “co-rotation” radius in barred potential.
• Causes “Fig-8” shape in rotation curve.

Kinematics of gas, in [NII]
[BM] Fig 4.60

ROUND BULGE  PEANUT-SHAPED

Stable orbits (stars)  Gas density  Gas velocity  Max. gas density
Bars appear to be easily excited instability in disks
Trailing vs. leading spirals
Which is the near side of the galaxy?
Molecular clouds on inner edges of arms

CO contours over red image

CO contours over 21 cm map

young star clusters

dark nebula/molecular clouds

HII regions

far side
dust lanes

approaching side

central bulge

receding side

near side

optical spiral arms
Swing Amplification

Position of (leading) spiral density enhancement

Epicyclic orbit of star

Automatically converts any leading spirals into much stronger trailing spirals.
Spiral Structure of the Milky Way

New model
- mix of
  - 2-armed mode
  - 4-armed mode
- Sun at ~ co-rotation radius.

Hard to measure, because we are inside it.

From HI (21 cm observations) assuming circular rotation.

Map of nearby young objects

N-body simulation

**FIG. 3**—Visible structure of the Galaxy derived for the best model (superposition of 2+4 self-sustained wave harmonics) by means of cloud-particle simulation. The scale is indicated in kpc. Note that the model is not valid for radii smaller than about 2.5 kpc.
Summary: Density Waves?

- Evidence showing density waves do occur.
  - Old, red stars show spiral density perturbation.
  - Molecular clouds form on inner edges of spiral arms.
  - HI gas flow shows discontinuity due to shocks at inner edges of spiral arms.
  - Bright young stars also in narrow arms.
    - Observed width $\Delta \theta \sim t_*(\Omega - \Omega_p)$, as predicted.
- Are these waves self-sustaining over $10^{10}$ years? Problems:
  - Lin-Shu theory is linear; does not predict whether waves will grow or decay.
  - How are density waves initially formed?
- The usual interpretation
  - Density waves need a driving force
    - Satellite galaxy at co-rotation radius (M51)
    - Bars
  - Otherwise, act to prolong life of transitory phenomena.
  - Other mechanisms probably also important.
    - Swing-amplification efficiently builds up temporary trailing spirals.