Worksheet #9 - PHY102 (Spr. 2004)

Due March 29th 2004

More on "Do loops", Illustrating Chaos

Tools that you need

You will need the following this week (look them up in the online help):
Do (you can also use Table or NestList)

You will also need to learn how to plot lists of numbers using:
ListPlot

In addition, you need to recall that animation is very simple in mathematica. Simply generate a series of frames (e.g. using a "Do" loop) and then double click on one of the frames. This automatically animates the set of frames.

The new physics - Chaos

Chaos, though discussed extensively for a couple of centuries (e.g. Boltzmann and Maxwell discussed "molecular chaos"), has really come into its own since the widespread use of computers. An early surprise is that even quite simple looking systems can have chaos, whereas it was originally thought that chaos only occured in systems with billions of molecules. In this worksheet you will study perhaps the simplest system which shows chaos, namely the "mapping"

$$x_{n+1} = \lambda x_n(1 - x_n). \tag{1}$$

This mapping models, for example, how a population density, x_{n+1}, changes as a function of the number of generations, n. Actually, it is not a very realistic model but it does illustrate many of the features of more complex systems. The parameter λ can be considered to be the "birth rate", ie. the number of offspring from the last generation. Anyway, the way it works is that if we know the population density at some time and call that density x_0, then the population density of the next generation is $x_1 = \lambda x_0(1 - x_0)$. This procedure is continued using Eq. 1 to find the population density for later generations. Intuitively, chaos means lack of order. Mathematically, it is defined by how stable the behavior of a set of equations is to small perturbations in the initial conditions. In the context of Equation 1 this means, how stable are the set of iterates $(x_0, x_1, x_2, x_3...)$ when you make the small change $x_0 \to x_0^\delta = x_0 + \delta x_0$. If this change is made, we get a new set of iterates $(x_0^\delta, x_1^\delta, x_2^\delta, x_3^\delta ...)$. If a set of equations is in a chaotic regime then the divergence of trajectories is exponential with a positive "Lyapunov" exponent. In the context of our example,

$$|x_n^\delta - x_n| \approx e^{\nu n}, \tag{2}$$

where, in a *truly chaotic* system, the Lyapunov exponent ν is positive.

Problem.

(i) Write a Mathematica code to iterate the mapping (Eq. 1). Plot the steady-state behavior of the mapping as a function of the parameter λ for $3 < \lambda < 4$.

(ii) In the regime which "looks chaotic" in your graph, obtain an estimate of the Lypunov exponent using Eq. 2.