Problem Set #1

PHY 854, Spring Semester, 2004
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These problems are due on March 19, 2003 5:00 p.m. to my mailbox on the first floor of BPS. Note, $\hbar = c = 1$. Feel free to work together, but each of you turn in a version and indicate who collaborated on what problems.

Problem 1 As was done in class, show that the two dimensional matrix representation for the $c$ element of $D_3$ is:

$$
\begin{pmatrix}
\frac{1}{2} & \frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{pmatrix}
$$

Problem 2 For $D_3$, show that $f(x, y) = xy$ is a basis function in the 2-dimensional representation.

Problem 3 Complete the formalism to show that the algebra of $SU(2)$ contains the term

$$[X_1, X_2] = -2X_3.$$  

Remember that the infinitesimal transformation on the vectors is

$$\delta \xi^i = \eta^{ij} \xi^j = U^i_\sigma \delta \alpha^\sigma,$$

that the infinitesimal transformation matrix is

$$\eta = \begin{pmatrix}
i \alpha^1 & \alpha^2 + i \alpha^3 & \alpha^3 - i \alpha^1 \\
-\alpha^2 + i \alpha^3 & -i \alpha^1 & \alpha^1 + i \alpha^3
\end{pmatrix},$$

and that the generators are defined as

$$X_\sigma = U^i_\sigma \frac{\partial}{\partial \xi^i}.$$  

Problem 4 (a) Show that the Lorentz character of

$$\bar{\psi}(x) \gamma^5 \psi(x)$$

is that of a scalar and that the parity character is that of a pseudoscalar.
Problem 5 Start with the semi-classical Hamiltonian density for the Dirac field:

\[ H = \int d^3x \psi_j^\dagger(x) (-i \alpha \cdot \nabla + \beta m)_{jk} \psi_k(x) \]

and the Fourier expansion of that field,

\[ \psi_j(x) = \sum_{i=1,2} \int dK \left[ a^{(i)}(k) u_j^{(i)}(k) e^{-ik \cdot x} + b^{(i)}(k) \nu_j^{(i)}(k) e^{ik \cdot x} \right] \]

where \( dK \equiv \frac{d^3k}{(2\pi)^3} \omega_k \). Quantize the field and show that the Hamiltonian operator becomes

\[ H = \int dK \sum_{i=1,2} E \left[ a^{(i)}(k) a^{(i)}(k) + b^{(i)}(k) b^{(i)}(k) \right]. \]

Problem 6 (a) Show that \( \frac{1}{2} (1 \pm \gamma_5) \) is a projection operator for right-handed (top sign) or left-handed (bottom sign) Dirac spinors.

(b) Define left-handed and right-handed components and show that the Lagrangian density for free spin 1/2 fields

\[ \mathcal{L}(x) = \bar{\psi}(x) \left( i \gamma^\mu \partial_\mu - m \right) \psi(x) \]

can be written

\[ \mathcal{L}(x) = \bar{\psi}_L(x) i \gamma^\mu \partial_\mu \psi_L(x) + \bar{\psi}_R(x) i \gamma^\mu \partial_\mu \psi_R(x) - m \left( \bar{\psi}_R(x) \psi_L(x) + \bar{\psi}_L(x) \psi_R(x) \right). \]

(c) Show that the Dirac Equation for \( \psi(x) \) is retrieved using the Euler Lagrange equations.

(d) Show that one can retrieve the Dirac equation for \( \bar{\psi}(x) \).

Problem 7 The Lagrange density for a particular important theory for a real scalar field can be written \( \lambda > 0 \)

\[ \mathcal{L}(x) = \frac{1}{2} \left( \partial_\mu \phi \right) \left( \partial^\mu \phi \right) - \frac{\lambda}{4} (\phi^2 - a)^2 \]

(a) What is the configuration of \( \phi(x) \) for which the classical energy is a minimum? Work it out for both cases of \( a > 0 \) and \( a < 0 \).

(b) If \( \lambda < 0 \) does the system have a ground state?

(c) Do you know why this particular theory is important in elementary particle physics? In a theory of critical phenomena?

Problem 8 The generic charge operator for spin 0 fields is

\[ T = \int dK \left[ a^\dagger(k) a(k) - b^\dagger(k) b(k) \right]. \]
As the generator of a conserved $U(1)$ transformation, this operator serves as both a constant of the motion and as the generator. Show that

$$\phi \rightarrow \phi' = U \phi U^{-1} = e^{-i\alpha} \phi$$

where

$$U = e^{i\alpha T},$$

Remembering that $T$ is an operator. In order to show this, you will have to work out the commutator of $T$ with $\phi$.

**Problem 9** Calculate the equation of motion for a massive spin 1 field, $B_\mu$. Start with the general relation that we got for the photon before the imposition of the Lorentz condition, namely,

$$\frac{\partial^2 B^\nu}{\partial x^\nu \partial x_\mu} - \frac{\partial}{\partial x_\nu} \left( \frac{\partial B^\mu}{\partial x^\nu} \right) = 0.$$

To switch to the massive case, we can replace the derivative operator

$$\frac{\partial^2}{\partial x^\nu \partial x_\mu} \rightarrow \frac{\partial^2}{\partial x^\nu \partial x_\mu} + M^2.$$

(a) Show that the relation

$$\left( \frac{\partial B^\mu}{\partial x^\nu} \right) = 0$$

then follows. Note, this is not a gauge condition, although it looks like it.

(b) Show that the equation of motion is

$$\left( \frac{\partial^2}{\partial x^\nu \partial x_\mu} + M^2 \right) B^\nu = 0.$$