Problem Set #1

PHY 854, Spring Semester, 2004

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These problems are due on March 19, 2003 5:00 p.m. to my mailbox on the first floor of BPS. Note, $\hbar = c = 1$. Feel free to work together, but each of you turn in a version and indicate who collaborated on what problems.

Problem 1 As was done in class, show that the two dimensional matrix representation for the c element of D_3 is:

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Problem 2 For D_3, show that $f(x, y) = xy$ is a basis function in the 2-dimensional representation.

Problem 3 Complete the formalism to show that that the algebra of $\mathsf{SU}(2)$ contains the term

$$[X_1, X_2] = -2X_3.$$

Remember that the infinitesimal transformation on the vectors is

$$\delta\xi^i = \eta^i_j \xi^j = U^i_\sigma \delta\alpha^\sigma,$$

that the infinitesimal transformation matrix is

$$\eta = \begin{pmatrix} i\alpha^1 & \alpha^2 + i\alpha^3 \\ -\alpha^2 + i\alpha^3 & -i\alpha^1 \end{pmatrix},$$

and that the generators are defined as

$$X_\sigma = U^i_\sigma \frac{\partial}{\partial \xi^i}.$$

Problem 4 (a) Show that the Lorentz character of

$$\bar{\psi}(x)\gamma^5\psi(x)$$

is that of a scalar and that the parity character is that of a pseudoscalar.

Problem 5 Start with the semi-classical Hamiltonian density for the Dirac field:

$$H = \int d^3x \psi_j^\dagger(x) \left(-i\boldsymbol{\alpha} \cdot \nabla + \beta m\right)_{jk} \psi_k(x)$$

and the Fourier expansion fof that field,

$$\psi_j(x) = \sum_{i=1,2} \int dK \left[a^{(i)}(k) u_j^{(i)}(k) e^{-ik\cdot x} + b^{\dagger(i)}(k) v_j^{(i)}(k) e^{ik\cdot x} \right]$$

where $dK \equiv \frac{d^3k}{(2\pi)^3 2\omega_k}$. Quantize the field and show that the Hamiltonian operator becomes

$$H = \int dK \sum_{i=1,2} E \left[a^{\dagger(i)}(k) a^{(i)}(k) + b^{\dagger(i)}(k) b^{(i)}(k) \right].$$

Problem 6

(a) Show that $\frac{1}{2}(1 \pm \gamma_5)$ is a projection operator for right-handed (top sign) or left-handed (bottom sign) Dirac spinors.

(b) Define left-handed and right-handed components and show that the Lagrangian density for free spin 1/2 fields

$$\mathcal{L}(x) = \bar{\psi}(x) \left(i\gamma^\mu \partial_\mu - m\right) \psi(x)$$

can be written

$$\mathcal{L}(x) = \bar{\psi}_L(x) i\gamma^\mu \partial_\mu \psi_L(x) + \bar{\psi}_R(x) i\gamma^\mu \partial_\mu \psi_R(x) - m \left(\bar{\psi}_R(x)\psi_L(x) + \bar{\psi}_L(x)\psi_R(x) \right).$$

(c) Show that the Dirac Equation for $\psi(x)$ is retrieved using the Euler Lagrange equations.

(d) Show that one can retrieve the Dirac equation for $\bar{\psi}(x)$.

Problem 7 The Lagrange density for a particular important theory for a real scalar field can be written ($\lambda > 0$

$$\mathcal{L}(x) = \frac{1}{2} \left(\partial_\mu \phi\right) \left(\partial^\mu \phi\right) - \frac{\lambda}{4} \left(\phi^2 - a\right)^2$$

(a) What is the configuration of $\phi(x)$ for which the classical energy is a minimum? Work it out for both cases of $a > 0$ and $a < 0$.

(b) If $\lambda < 0$ does the system have a ground state?

(c) Do you know why this particular theory is important in elementary particle physics? In a theory of critical phenomena?

Problem 8 The generic charge operator for spin 0 fields is

$$T = \int dK \left[a^\dagger(k) a(k) - b^\dagger(k) b(k) \right].$$

As the generator of a conserved $U(1)$ transformation, this operator serves as both a constant of the motion and as the generator. Show that

$$\begin{aligned}\phi \rightarrow \phi' =& U\phi U^{-1} \\ =& e^{-i\alpha}\phi\end{aligned}$$

where

$$U = e^{i\alpha T},$$

Remembering that T is an operator. In order to show this, you will have to work out the commutator of T with ϕ.

Problem 9 Calculate the equation of motion for a massive spin 1 field, B_μ. Start with the general relation that we got for the photon before the imposition of the Lorentz condition, namely,

$$\frac{\partial^2 B^\nu}{\partial x^\mu \partial x_\mu} - \frac{\partial}{\partial x_\nu}\left(\frac{\partial B^\mu}{\partial x^\mu}\right) = 0.$$

To switch to the massive case, we can replace the derivative operator

$$\frac{\partial^2}{\partial x^\mu \partial x_\mu} \rightarrow \frac{\partial^2}{\partial x^\mu \partial x_\mu} + M^2.$$

(a) Show that the relation

$$\left(\frac{\partial B^\mu}{\partial x^\mu}\right) = 0$$

then follows. Note, this is not a gauge condition, although it looks like it.

(b) Show that the equation of motion is

$$\left(\frac{\partial^2}{\partial x^\mu \partial x_\mu} + M^2\right) B^\nu = 0.$$