Lecture 4 1/20/04 Continuous Bases

functional bases: $\in \mathbb{L}^2_1(a,b)$ can be constructed which could be countably infinite:

could be orthonormal : $(\cancel{\eta^{(i)}}$ $$(\eta^{(i)}, \eta_{(j)}) = \int_a^b \eta^{i*}(x)\eta_j(x)\,dx$$

$$\langle \eta^{(i)} | \eta_{(j)} \rangle = \delta^i{}_j$$

expand – "spectral decomposition"

$$\xi(x) = \sum_{i=1}^{\infty} \eta_{(i)}(x)\, \xi^i_{[\eta]}(x) \qquad \star$$

countably ∞

One can imagine forming the "best" representation of a vector $\in \mathbb{L}^2_1(a,b)$. – in the sense of the RHS of $\star$ converging to the LHS of $\star$.

Form 'mean free error'

$$M = \int_a^b \left| \xi(x) - \sum_{i=1}^{n} \eta_{(i)}(x)\, \xi^i_{[\eta]} \right|^2 dx \qquad \text{drop } [\eta]$$

$$= \int_a^b |\xi(x)|^2 - \int_a^b \sum_i \xi^*(x)\,\eta_{(i)}(x)\,\xi^i\,dx$$

$$- \int_a^b \sum_i \eta^{(i)*}(x)\,\xi_i^*(x)\,dx$$

$$+ \int \sum\sum \eta^*_{(j)}(x)\,\eta_{(i)}(x)\,\xi^{*j}\,\xi_i\,dx$$

$$= N_\xi - \sum_i \left[(\xi, \eta_{(i)})\xi^i + (\eta^{(i)}, \xi)\xi_i^* - \xi_i^*\xi^i \right]$$

find extremum

$$\frac{dM}{d\xi^*} = 0 = (\eta^{(i)}, \xi) - \xi^i$$

so $\xi^i = (\eta^{(i)}, \xi)$

and coefficients are best represented → like "best fit"

$$\xi^i = \int_a^b \eta^{*(i)}(x)\,\xi(x)\,dx \qquad \text{familiar.}$$

$\boxed{XI}$. Completeness, essentially says that $M \to 0$ as $n \to \infty$

or

$$\lim_{n\to\infty} \int_a^b \left| \xi - \sum_{i=1}^{n} \eta_{(i)}\, \xi^i \right|^2 dx \to 0 \qquad \text{"in a Cauchy sense"}$$

I, II, III, V, X, XI constitute the definition of a Hilbert space

$$\xi(x) = \sum_{i=1}^{n} \eta_{(i)}(x)\, \xi^i_{[n]}$$

$$= \sum_i \eta_{(i)}(x) \int_a^b \eta^{*(i)}(x')\, \xi(x')\, dx'$$

$$= \int_a^b \underbrace{\left\{ \sum_i \eta_{(i)}(x)\, \eta^{*(i)}(x') \right\}}_{\equiv\, f(x,x')} \xi(x')\, dx'$$

$$\xi(x) = \int_a^b f(x,x')\,\xi(x')\,dx' \qquad \text{defines the Dirac } \delta \text{ function.}$$

Completeness/Closure: $\displaystyle\sum_{i=1}^{\infty} \eta_{(i)}(x)\,\eta^{*j}(x') = \delta(x-x')$

The whole field came about as a series of lectures by Hilbert in 1925-1927... von Neumann was his post doc.

functional analysis $\longleftrightarrow$ ~~vector~~ linear algebra.

Named Hilbert Space by Courant in famous books (of lecture series)

Most of the famous square-integrable functions in physics are basis functions in a Hilbert Space — H_n, P_n, C_n, L_n - etc.

Many are not square-integrable as they are defined over $[-\infty, +\infty] \longrightarrow$ integral transforms. $\int_a^b$ not defined.

like Fourier:

$$g(p) = \sqrt{1/2\pi} \int_{-\infty}^{\infty} dx\, f(x)\, e^{-ipx/\hbar}$$

$$f(x) = \sqrt{1/2\pi} \int_{-\infty}^{\infty} dp\, g(p)\, e^{ipx/\hbar}$$

like Σ

"basis": $\eta_p(x) = \sqrt{1/2\pi}\, e^{-ipx/\hbar}$

so,

$$g(p) = \int_{-\infty}^{\infty} dx\, \eta_p(x) f(x)$$

$$f(x) = \int_{-\infty}^{\infty} dp\, \eta_p^*(x)\, g(p)$$

$$\sqrt{1/2\pi} \int_{-\infty}^{\infty} \eta_p(x)\, dx = \delta(x)$$

Completeness $\int dp\, \eta_p(x)\, \eta_p^*(x) = \int \frac{dp}{2\pi} e^{-ip(x-x')/\hbar} \equiv \delta(x-x')$

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So, notation for 'function vectors.'

Discrete basis: $|\xi(x)\rangle = \sum_{i=1}^{\infty} |\eta_{(i)}(x)\rangle \, \xi^{i}_{[\eta]}$

Continuum basis: $|\xi(x)\rangle = \int_{\Omega_{x\omega}} d\omega \, |\gamma_x(\omega)\rangle \, \xi_{[\gamma]}(\omega)$

and for continuum basis:

A Completeness: $\int_{\Omega_x} dx \, |\gamma_x(\omega)\rangle\langle\gamma_x(\omega')| = \delta(\omega-\omega')$

B Orthonormality: $\langle \gamma_x(\omega) | \gamma_x(\omega')\rangle = \delta(\omega-\omega')$

and discrete basis:

A Completeness: $\sum_{i=1}^{\infty} |\eta_{(i)}(x)\rangle\langle \eta^{(i)}(x')| = \delta(x-x')$

B Orthonormality: $\langle \eta^{(j)} | \eta_{(i)}\rangle = \delta^{j}{}_{i}$

Consider a basis function $\eta_p(x)$

$$\eta_p(x) = \sqrt{1/2\pi} \; e^{ipx}$$

Assign a Hilbert vector to it $|p\rangle$ such that

$$\langle p'|p\rangle = \int dx \; \eta^{*}_{p'}(x)\, \eta_p(x) \quad \left(= \langle \eta_{p'}(x) | \eta_p(x)\rangle \right)$$

$$= \int dx \, e^{-ip'x} e^{ipx} \left(\tfrac{1}{2\pi}\right) \equiv \delta(p'-p)$$

Can pass functions between canonically conjugate quantities using $|\eta_p(x)\rangle$

$$\phi(x) = \int dp\, \tilde{\Phi}(p)\, \eta_p(x)$$

$$\tilde{\Phi}(p) = \int dx\, \phi(x)\, \eta_p^*(x)$$

Expand an arbitrary state vector:

$$|\xi\rangle = \int dp'\, |p'\rangle\, \tilde{\xi}(p') \qquad \text{spectral expansion}$$

from $\qquad \langle p|\xi\rangle = \int dp'\, \langle p|p'\rangle\, \tilde{\xi}(p')$

$$\underbrace{\langle p|\xi\rangle}_{} = \int dp'\, \delta(p-p')\, \tilde{\xi}(p') = \tilde{\xi}(p)$$

projection ~~of arbitrary~~ abstract ξ "along" a basis function $|p\rangle$ ← H.S.

get function of p

So,

$$|\xi\rangle = \int dp'\, |p'\rangle\langle p'|\xi\rangle$$

(like before $\quad |\xi\rangle = |e_k\rangle\langle e^k|\xi\rangle$)

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Do it again -- another continuum basis $\delta_{x'}(x) = \delta(x-x')$
w) H.S. vector assignment $|x'\rangle$

normalize $\quad \langle x''|x'\rangle = \int dx\, \delta^*_{x''}(x)\, \delta_{x'}(x)$

$$= \int dx\, \delta(x-x'')\,\delta(x-x')$$

$$= \delta(x''-x') \Rightarrow \text{no existing}$$

norm --
not square
integrable

Can still expand

some function.

$$|\xi\rangle = \int dx'\, |x'\rangle F_\xi(x')$$

$\langle p|\xi\rangle = \int dx'\, \langle p|x'\rangle F_\xi(x') = \hat{\xi}(p)$ from before.

so $\quad \langle p|x'\rangle = \int dx\, \eta_p^*(x)\, \delta_{x'}(x)$

$$= \int dx\, \eta_p^*(x)\, \delta(x-x')$$

$$= \eta_p^*(x') \quad \text{or} \quad \langle x|p\rangle = \eta_p(x)$$

So .. continuing

$$\langle p|\xi\rangle = \int dx'\, \eta_p^*(x')\, F_\xi(x') \overset{\text{still}}{=} \hat{\xi}(p)$$

so ... $F_\xi(x')$ is the F.T. of $\hat{\xi}$ or. $\Rightarrow \xi(x')$

<u>Now</u> $\quad |\xi\rangle = \int dx'\, |x'\rangle\, \xi(x')$

expanding in terms of the configuration basis

$$\langle x|\xi\rangle = \int dx' \langle x|x'\rangle \xi(x')$$
$$= \int dx' \, \delta(x-x')\, \xi(x')$$

$$\langle x|\xi\rangle = \xi(x)$$

projection of abstract H.S. vector, ξ, along configuration space basis $\longrightarrow$ FUNCTION OF x.

$\langle x|\xi\rangle$ = wavefunction in coordinate space

$\langle p|\xi\rangle$ = wavefunction in momentum space.

$\longrightarrow$ The essence of Dirac's Representation Theory... how he connected the discrete energy-transition description of Heisenberg to the continuous picture of Schrödinger

An important part of the story is the linking of dynamical variables to suitable operators.

Copenhagen view about how state vectors can be transformed:

1) continuously, through dynamics.

2) discontinuously, through measurement.

1) is understood to be a mapping $|\xi\rangle \xrightarrow{M} |\alpha\rangle$

ie $M|\xi\rangle = |\alpha\rangle$ → always linear operators in QM. XII.

XIII. Operator self-adjoint or Hermitian if

$$M = M^{\dagger} \quad \text{where} \quad (M^{\dagger})^{i}{}_{j} = (M^{j}{}_{i})^{*}$$

and unitary if $MM^{\dagger} = M^{\dagger}M = \mathbb{1}$

XIV. For the following relationship

$$A|\alpha_{(i)}\rangle = |\alpha_{(j)}\rangle \delta^{j}{}_{i}\, a(i) \quad \rightarrow \text{no sum.}$$

eigenvalue of i^{th} eigenvector.

$$= |\alpha_{(i)}\rangle a(i)$$

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A matrix representation for linear operators is efficacious —

Consider: $|\alpha\rangle = \underline{\underline{|\eta_{(i)}\rangle \alpha^i_{[\eta]}}}$

$|\beta\rangle = \underline{|\eta_{(j)}\rangle \beta^j_{[\eta]}}$

2 vectors, same basis.

Connect them through a mapping —

$$|\alpha\rangle \rightarrow |\beta\rangle = M|\alpha\rangle$$

$$\underline{|\eta_{(j)}\rangle \beta^j_{[\eta]}} = \underline{\underline{M|\eta_i\rangle}} \alpha^i_{[\eta]}$$

from $\langle \eta^k | \underline{\beta}\rangle = \underbrace{\langle \eta^k | M | \eta_i\rangle}_{} \alpha^i_{[\eta]}$ $\ast$

a complex number depending on 2 indices → a matrix.

$$\langle \eta^k | \eta_j\rangle \beta^j_{[\eta]} =$$

$$\delta^k_j \, \beta^j_{[\eta]} = \beta^k_{[\eta]} = M^k{}_i \alpha^i_{[\eta]}$$

This defines the "matrix element" $M^k{}_i = \langle \eta^k | M | \eta_i\rangle$

an abstract object

a representative of M in η basis.

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How about for a continuous basis.

$$\begin{array}{ccccc} & & C & & D \\ |\xi\rangle & = & \int d\alpha\, |\gamma(\alpha)\rangle\, \xi_{[\gamma]}(\alpha) & = & |\eta_i\rangle\, \xi^i_{[\eta]} \\ |\rho\rangle & = & \int d\beta\, |\gamma(\beta)\rangle\, \rho_{[\gamma]}(\beta) & = & |\eta_j\rangle\, \rho^j_{[\eta]} \end{array}$$

completeness

D: $$\langle\rho|M|\xi\rangle = \sum_i\sum_j \langle\rho|\eta_i\rangle\langle\eta^i|M|\eta_j\rangle\langle\eta^j|\xi\rangle$$

$$= \sum_i\sum_j \rho^*_{i[\eta]}\, M^i_{[\eta]j}\, \xi^j_{[\eta]}$$

C: $$\langle\rho|M|\xi\rangle = \int d\alpha \int d\beta\, \langle\rho|\gamma(\alpha)\rangle\langle\gamma(\alpha)|M|\gamma(\beta)\rangle\langle\gamma(\beta)|\xi\rangle$$

$$= \int d\alpha \int d\beta\, \rho^*_{[\gamma]}(\alpha)\, \xi_{[\gamma]}(\beta)\langle\gamma(\alpha)|M|\gamma(\beta)\rangle$$

XV. Products of operators $\longrightarrow$ other operators.

$$A|\eta_i\rangle = |\eta_j\rangle A^j{}_i$$
$$B|\eta_n\rangle = |\eta_k\rangle B^k{}_n$$

$$AB|\eta_n\rangle = (A|\eta_k\rangle) B^k{}_n = |\eta_j\rangle \underbrace{A^j{}_k\, B^k{}_n}$$

arranged for matrix multiplication $\sum_k$.

$$AB|\eta_n\rangle = |\eta_j\rangle (AB)^j{}_n$$

note: $$BA|\eta_n\rangle = B|\eta_k\rangle A^k{}_n = |\eta_m\rangle B^m{}_k A^k{}_n$$
$$= |\eta_m\rangle (BA)^m{}_n$$

not necessarily $= AB$

Consider another <u>change of basis.</u> - 2 different bases.

$$|\gamma_i\rangle = |\eta_k\rangle M^k{}_i$$

$$|\eta_i\rangle = |\gamma_j\rangle (M^{-1})^j{}_i$$

and an single arbitrary operator. $A|\gamma_i\rangle = |\gamma_n\rangle A^n_{[\gamma]\,i}$

$$A|\eta_m\rangle = |\eta_p\rangle A^p_{[\eta]\,m}$$

How are ~~they related~~ the representatives $A_{[\gamma]}$ and ~~Ex~~ $A_{[\eta]}$ related?

$$A|\gamma_i\rangle = |\gamma_n\rangle A^n_{[\gamma]\,i} = |\eta_k\rangle M^k{}_n A^n_{[\gamma]\,i}$$

and

$$A|\gamma_i\rangle = A|\eta_j\rangle M^j{}_i = |\eta_p\rangle A^p_{[\eta]\,j} M^j{}_i$$

so, for a given k $\quad M^k{}_n A^n_{[\gamma]\,i} = A^k_{[\eta]\,j} M^j{}_i$

since bases are O.N., and M^{-1} exists.

PROBLEM

$$(M^{-1})^\ell{}_k M^k{}_n = \delta^\ell{}_n$$

$$\delta^\ell{}_n A^n{}_{(\gamma)\,i} = (M^{-1})^\ell{}_k A^k_{(\eta)\,j} M^j{}_i$$

$$\longrightarrow \quad A^\ell_{[\gamma]\,i} = (M^{-1})^\ell{}_k A^k_{[\eta]\,j} M^j{}_i$$

or $\quad \mathbb{A}_{(\gamma)} = \mathbb{M}^{-1} \mathbb{A}_{(\eta)} \mathbb{M} = \mathbb{M}^\dagger \mathbb{A}_{(\eta)} \mathbb{M}$

a <u>similarity</u> <u>Transformation</u>

Leave bases alone and transform the vectors.

$$|\beta\rangle = O|\alpha\rangle$$

$$A|\alpha\rangle = |\alpha'\rangle$$
$$A|\beta\rangle = |\beta'\rangle$$

(what operator maps $\alpha' \to \beta'$?

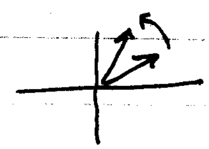
$$|\alpha\rangle = A^{-1}|\alpha'\rangle$$

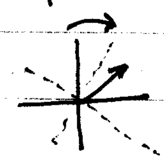
$$|\beta'\rangle = A|\beta\rangle = AO|\alpha\rangle$$
$$= (AOA^{-1})|\alpha'\rangle = O'|\alpha'\rangle$$

so

$$O' = AOA^{-1}$$ opposite sense of similarity transformation.

→ always confusing...
the issue between

Active rotations 

Passive rotations 

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Dynamical quantities are continuous parameters..

$\downarrow$ assumption of QM

operators.

remember.

D: $$A|\eta_i\rangle = |\eta_j\rangle \delta^j{}_i \, \alpha(i)_{[\eta]} \quad \text{no sum.}$$

$$\langle \eta^k | A | \eta_i \rangle = \langle \eta^k | \eta_j \rangle \delta^j{}_i \, \alpha(i)_{(\eta)}$$

$$A^k{}_i = \delta^k{}_j \delta^j{}_i \, \alpha(i)$$

$$\rightarrow \delta^i{}_k A^k{}_i = \delta^i{}_k \delta^k{}_j \delta^j{}_i \, \alpha(i)$$
$$= \delta^i{}_j \delta^j{}_i \, \alpha(i)$$
$$A^i{}_i = \alpha(i) \rightarrow \text{diagonal.}$$

C: coordinate rep. $\langle x | \xi \rangle = \psi_\xi(x)$ coord w.f.

$\downarrow$ H.S. state vector

$|x\rangle$

operator related to x.

$$\underline{X} \, | \psi_\xi(x) \rangle = | \psi_\xi(x) \rangle \, x \quad \text{or}$$

$$\underline{X} | x \rangle = | x \rangle \, x$$

$$\langle x' | \underline{X} | x \rangle = \langle x' | x \rangle \, x = \underbrace{\delta(x' - x)}_{\text{"diagonal"}} \, x$$

like above.

Likewise $$\underline{P} \, | \psi_\xi(p) \rangle = | \psi_\xi(p) \rangle \, p$$
$$\underline{P} | p \rangle = | p \rangle \, p$$

and have $\langle p' | p \rangle = \delta(p - p')$

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Consider abstract state vector $|\xi\rangle$

completeness —

$$\underline{P}|\xi\rangle = \int dx\, \underline{P}|x\rangle\langle x|\xi\rangle$$

$$= \int dx\, \underline{P}|x\rangle \psi_\xi(x)$$

$$\langle x'|\underline{P}|\xi\rangle = \int dx\, \underbrace{\langle x'|\underline{P}|x\rangle}_{\text{what's this...}} \psi_\xi(x)$$

$$\langle x'|\underline{P}|x\rangle = \int dp\, \langle x'|\underline{P}|p\rangle\langle p|x\rangle$$

$$= \int dp\, p\, \langle x'|p\rangle\langle p|x\rangle$$

since $\langle x|p\rangle = \psi_p(x)$
$\langle p|x'\rangle = \psi_p^*(x')$

$$\langle x'|\underline{P}|x\rangle = \int dp\, p\, \frac{e^{-ip(x'-x)}}{2\pi} \qquad \text{note}$$

diff of δ:

$$\frac{d}{dx}\delta(x'-x) = -i\int dp\, \frac{p}{2\pi}\, e^{-ip(x'-x)}$$

so

$$\langle x'|\underline{P}|x\rangle = +i\frac{d}{dx}\delta(x'-x) \equiv \delta'(x'-x)$$

So, $\langle x'|\underline{P}|x\rangle = \int dx\, i\, \delta'(x'-x)\, \psi_\xi(x)$

integrate by parts $\displaystyle\int_{-\infty}^{\infty} f(x)\,\delta'(x)\,dx = \underbrace{f(x)\delta(x)\Big|_{-\infty}^{\infty}}_{0} - \underbrace{\int f'(x)\,\delta(x)\,dx}_{f'(0)}$

again

So $$\langle x' | \underline{P} | \xi \rangle = i \int dx\, \delta'(x'-x)\, \psi_\xi(x)$$

$$= -i\, \psi_\xi'(x')$$

$$= -i \frac{d}{dx} \psi_\xi(x')$$

So the representation of the momentum operator in the coordinate representation is

$$\underline{P} = -i \frac{d}{dx}$$

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Back to group theory, now armed with linear algebra

<u>def</u>. A representation of a group $\mathcal{G}$ is a mapping of the elements of $\mathcal{G}$ onto a group of <u>Linear Operators</u> defined in a linear vector space, V.

$$\begin{array}{ccccc} \mathcal{G} = \{ & e, & a, & b & \ldots \} \\ & \downarrow & \downarrow & \downarrow & \\ \mathcal{G} = \{ & \Gamma(e), & \Gamma(a), & \Gamma(b), & \ldots \} \end{array}$$

map is <u>faithful</u>

So, if we have $a \circ b = c \Rightarrow \Gamma(a) \otimes \Gamma(b) = \Gamma(c)$ under whatever multiplication rule is operative for V.

Now, we can provide a mathematical realization for the Δ.. namely basis vectors on which the Γ within V.

Transformation properties:

$$\underline{|x\rangle \longrightarrow |x'\rangle = \Gamma|x\rangle}$$

~~arbitrary~~ & abstract.

Let $|x'\rangle = |e'_j\rangle x^{j'}$ $|x\rangle = |e_i\rangle x^i$

So $|x'\rangle = \Gamma|e_i\rangle x^i$

$\cancel{|e'_j\rangle x^{j'} = |}$

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again

$$|x'\rangle = \Gamma|e_i\rangle x^i = |e_j'\rangle x^{j'}$$

$$= |e_j\rangle \Gamma^j{}_i x^i$$

which implies a matrix representation for Γ

So, the components are $\underline{\underline{x^{j'} = \Gamma^j{}_i x^i}}$

$$\begin{pmatrix} x^{1'} \\ x^{2'} \\ \vdots \end{pmatrix} = \begin{pmatrix} \Gamma^1{}_1 & \Gamma^1{}_2 & \cdots \\ \vdots & & \\ \vdots & & \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ \vdots \end{pmatrix}$$

Usually the row/column designation is for components -- but also can be needed for bases.

$$|e\rangle = (e_1, e_2 \ldots)$$

Then,

$$|x\rangle = |e_{(i)}\rangle x^i \quad \text{can be written}$$

$$|x\rangle = \vec{x} = \overbrace{(|e_1\rangle |e_2\rangle |e_3\rangle)}^{\text{vectn}} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \Big\} \text{ another vectn.}$$

$\downarrow$ one object.

$$= \cancel{e_1 x^1 + e_2 x^2 + e_3 x^3}$$

$$= |e_{(j)}\rangle e^j{}_1 x^1 + |e_{(j)}\rangle e^j{}_2 x^2$$

The Γ can also be used to transform the <u>bases</u> ... the coordinate system.

$$|e_i\rangle \to |e_j'\rangle = \Gamma|e_i\rangle = |e_j\rangle \Gamma^j{}_i \quad \text{coordinates}$$

So,

$$|x'\rangle = |e_j\rangle x^{j'} = |e_j\rangle \underline{\underline{\Gamma^j{}_i x^i}}$$

$$= |e_i'\rangle x^i$$

} components 2 different descriptions.

notice that the locations of the $\Gamma^i{}_j$ are slightly different from what one expects for basis transformation

$$|e_1'\rangle = |e_1\rangle \Gamma^1{}_1 + |e_2\rangle \Gamma^2{}_1$$
$$|e_2'\rangle = |e_1\rangle \Gamma^1{}_2 + |e_2\rangle \Gamma^2{}_2$$

→ off-diagonal reversed from the transformation of the vector, rather than coordinate system. Difference is between ↶ and ↷ of active vs. passive rotation

$$|x'\rangle = \Gamma |e_j\rangle x^j = |e_j\rangle \Gamma^j{}_i x^i$$

so

$$x^{1'} = \Gamma^1{}_1 x^1 + \Gamma^1{}_2 x^2$$
$$x^{2'} = \Gamma^2{}_1 x^1 + \Gamma^2{}_2 x^2$$

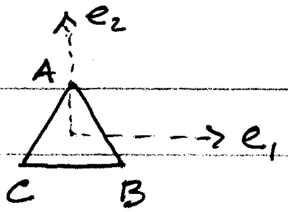
The elements of Γ can be formed by the inner product

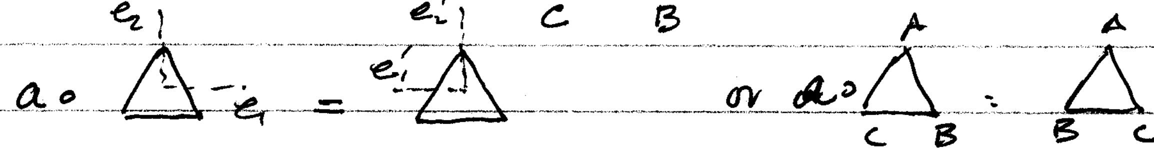
$$|e_i'\rangle = |e_j\rangle \Gamma^j{}_i$$
$$\langle e^k | e_i' \rangle = \langle e^k | e_j \rangle \Gamma^j{}_i$$
$$= \delta^k{}_j \Gamma^j{}_i = \Gamma^k{}_i$$

$$|e'\rangle = \Gamma |e\rangle$$
$$\langle e | e' \rangle = \langle e | \Gamma | e \rangle \quad \text{so} \quad \Gamma^k{}_i = \langle e^k | \Gamma | e_i \rangle$$

more specifically, $\Gamma^k{}_i(g) = \langle e^k | \Gamma(g) | e_i \rangle$

So... back to D_3





or. $e_1 \to e_1' = -e_1$, $\Gamma(a)\, e_1 = -e_1$
$e_2 \to e_2' = e_2$, $\Gamma(a)\, e_2 = e_2$

calculate them

$$\Gamma(a)^1{}_1 = \langle e^1 | \Gamma(a) | e_1 \rangle = -\langle e^1 | e_1 \rangle = -1$$

$$\Gamma(a)^1{}_2 = \langle e^1 | \Gamma(a) | e_2 \rangle = \langle e^1 | e_2 \rangle = 0 = \Gamma(a)^2{}_1$$

$$\Gamma(a)^2{}_2 = \langle e^2 | \Gamma(a) | e_2 \rangle = \langle e^2 | e_2 \rangle = 1$$

So, we have $\Gamma^{(2)}(a) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

which is a 2 dimensional representation of D_3.

So, $|e'\rangle = \Gamma(a)\, |e\rangle$ becomes.

$$\cancel{(e_1, e_2)} =$$
$$|e_i'\rangle = |e_j\rangle \Gamma^j{}_i(a)$$

$$= (e_1, e_2) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = (-e_1, e_2) = (e_1', e_2')$$

The others follow similarly.

PROBLEM → rest of 2d and 1d.
→ IRR example