1/12/04 lecture 1 Groups, Introduction

Introduction – plans, – schedule

× Classtime – need 2 periods of 75 min – regular
1 period of 75 min – extraordinary.

× Topics:

→ Broadly, Relativistic Quantum Field Theory.
But, to a purpose: modern day elementary particle physics.

Features that characterize modern field theories and models:

- use of Gauge Symmetry is primary.
- local field theories

Specifically:

- Lie Groups
- Classical Field Theory
- "Second Quantized" Quantum Field Theory
- S-Matrix and Covariant Perturbation Theory
- Examples from QED and Weak Interactions
- Renormalization: Pauli-Villars & Dimensional Regularization
- Weinberg-Salam approach to Gauge Theories

Group Theory in general.

The origins of group theory are rather obtuse and complicated. While its birth goes back before Lagrange, it really wasn't recognized in its modern form until the early 20^{th} century. Its antecedents come from

a) algebra – Galois and Abel
b) number theory – Cayley.
c) geometry – Klein

$\Big\}$ ~ 1900 – "group theory" postulates.

def: A group is a set of abstract elements $g \in \{a, b, c \dots\}$ for which there is a single composition law, $\circ$, (normally called "multiplication") which satisfies the following 4 properties:

1. If a and $b \in \mathcal{G}$ and $c = a \circ b$, then $c \in \mathcal{G}$ (closure)
2. $(a \circ b) \circ c = a \circ (b \circ c)$ (associativity)
3. $\exists\, g_i \equiv e \ni e \circ g_i = g_i \circ e = g_i$ (identity)
4. $\forall g_i \in \mathcal{G} \;\exists\; g' \ni g \circ g' = g' \circ g = e$ (inverse)

– THAT'S IT –

→ exactly like the definitions for a Vector Space*, except there is no commutivity. This was interesting – the power that resulted in taking a well-established algebraic structure – and removing a postulate.

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* Grassman. 1844 → abused and forgotten until rediscovered by Peano in 1880's.

order: The number of elements in a group, finite or infinite

Abelian Group: in which all elements commute

Discrete Group: a countable, finite order.

There are only a few Abelian Groups.

① $\{e\}$ order 1

② $\{e, a\}$ order 2 $\Rightarrow$ $e \circ a = a$

$e \circ e = e$

$a \circ a = e$

Conventional and standard way to represent: "multiplication table"

	e	a
e	e	a
a	a	e

Cyclic Group, C_2 (or the one, C_1)

③ $\{e, d, f\}$ order 3, C_3

	e	d	f
e	e	d	f
d	d	f	e
f	f	e	d

④ $\{e, a, b, c\}$

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Groups are abstract things.
I define a "Group Realization" and the more conventional "representation" . . .

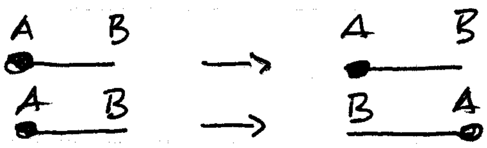
A realization for C_2 might be:

– $a = -1$, $e = +1$ w/ regular "multiplication"

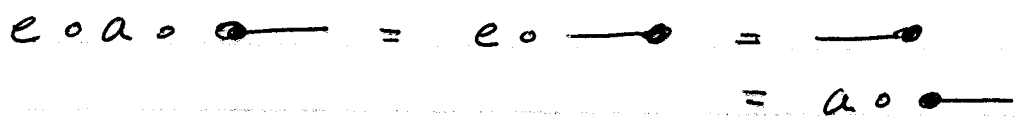
– geometrical

$e \equiv$ operation $e \circ$ … "let it be"

$a \equiv$ operation $a \circ$ … "flip it" "reflect"



Operations – the multiplication table – holds –
eg.



so $e \circ a = a$ etc.

The algebra involves the operations, not the matchsticks.

connection w/ QM.

Putting labels on the ends: e takes $A \to A$, $B \to B$

a takes $A \to B$, $B \to A$

So, $\quad a \circ a \circ \overset{A \qquad B}{\bullet\!\!-\!\!-\!\!-\!\!-} = a \circ \overset{B \qquad A}{-\!\!-\!\!-\!\!-\!\!\bullet} = \overset{A \qquad B}{\bullet\!\!-\!\!-\!\!-\!\!-}$

$$= a \circ \begin{pmatrix} A \to B \\ B \to A \end{pmatrix} = \begin{pmatrix} B \to A \\ A \to B \end{pmatrix}$$

This is another "realization" of C_2, namely permutations. The standard notation was invented by Cayley

element in LH slot (1^{st})

elements in RH slot (2^{nd})

$$\begin{pmatrix} L & R \\ i & j \end{pmatrix}$$

element in 2^{nd} goes to j^{th}

element in 1^{st} goes to i^{th}

So, the realization can be

$$e = \begin{pmatrix} L & R \\ L & R \end{pmatrix} \qquad a = \begin{pmatrix} L & R \\ R & L \end{pmatrix}$$

This notation implies $\quad \begin{pmatrix} L & R \\ L & R \end{pmatrix} = \begin{pmatrix} R & L \\ R & L \end{pmatrix}$

$$\begin{pmatrix} L & R \\ R & L \end{pmatrix} = \begin{pmatrix} R & L \\ L & R \end{pmatrix}$$

that is $\quad \begin{pmatrix} a & b & c \ldots \\ \alpha & \beta & \gamma \ldots \end{pmatrix} = \begin{pmatrix} b & a & c \ldots \\ \beta & \alpha & \gamma \ldots \end{pmatrix}$

$$a \circ e = \begin{pmatrix} L & R \\ R & L \end{pmatrix}\begin{pmatrix} L & R \\ L & R \end{pmatrix} = \begin{pmatrix} L & R \\ R & L \end{pmatrix} = a$$

$$a \circ a = \begin{pmatrix} L & R \\ R & L \end{pmatrix}\begin{pmatrix} L & R \\ R & L \end{pmatrix} = \begin{pmatrix} L & R \\ L & R \end{pmatrix} = e$$

$$\begin{pmatrix} R & L \\ L & R \end{pmatrix}\begin{pmatrix} L & R \\ R & L \end{pmatrix} = \begin{pmatrix} L & R \\ L & R \end{pmatrix}$$

arrange to make this so

This is the beginnings of a big subject – the Permutation or Symmetric Group on n objects, S_n. This group has $n!$ elements.

→ the real beginning of Group Theory.

Notice that C_2 is the "same" (⇒ same algebra) as S_2

<u>isomorphic</u> groups: groups having the same multiplication table & 1:1 mapping between elements.

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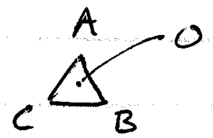
A geometrical representation of C_3 is also possible —



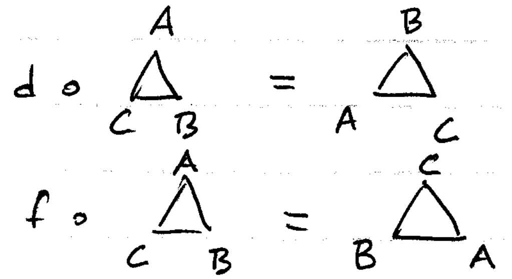
An equilateral triangle

e: nothing

d: $2\pi/3$ ↺ about O

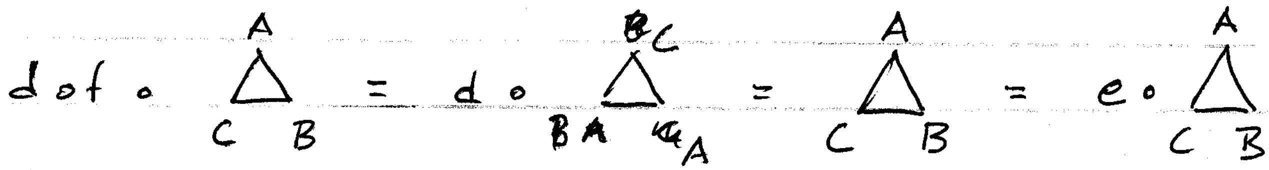
f: $4\pi/3$ ↺ $\equiv \frac{2\pi}{3}$ ↻ about O

This does not replace adjacent numbers. —



$$\begin{matrix} A & B & C \\ \downarrow & \downarrow & \downarrow \\ B & C & A \end{matrix}$$

$$\begin{matrix} A & B & C \\ \downarrow & \downarrow & \downarrow \\ C & A & B \end{matrix}$$



so $d \circ f = e$, cyclic replacement.

There is a unit of permutation here too —

Write $$d = \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}$$

$$f = \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}$$

$$d \circ f = \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix} \circ \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}$$

$$= \begin{pmatrix} C & A & B \\ A & B & C \end{pmatrix} \circ \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix} = \begin{pmatrix} C & A & B \\ C & A & B \end{pmatrix} = e$$

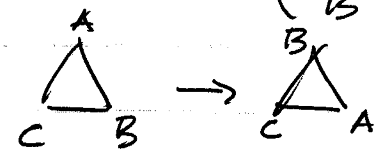
Is C_3 the same thing as S_3? No.

\# elements in $S_3 = 3! = 6$

\# elements in $C_3 = 3$

also, S_3 would have elements like $\begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$

which C_3 does not do.



We can collect all the elements of S_3 -- all permutations on 3 things:

$$e = \begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix}, \quad a = \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix} \quad b = \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$$

$$c = \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix}$$

$$d = \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix} \quad f = \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix}$$

<u>subgroups</u>: A subset of elements, $\mathcal{H}$ from a group of elements $\mathcal{G}$ which itself forms a group having the same combinatoric law of $\mathcal{G}$.

Notice: $\{e, a\}$, $\{e, b\}$, $\{e, c\}$ form 3 C_2 subgroups of S_3 (or S_2 subgroups)

and

$\{e, d, f\}$ is a single C_3 subgroup of S_3.

called the alternating group A_n

A_n corresponding to S_n has $n!/2$ elements

Notice also, in S_3:

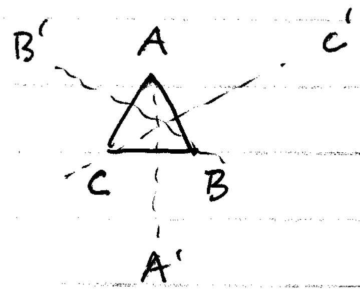
$$a \circ b = \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix} \circ \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix}$$

$$= \begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix} \cdot \begin{pmatrix} A & C & B \\ C & A & B \end{pmatrix} = \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix} = f$$

$b \circ a = d \quad \Rightarrow \quad a \circ b \neq b \circ c \Rightarrow$ non Abelian

C_3 is smallest non-Abelian g.

Most familiar realization of S_3 is the "covering group" of an equilateral triangle.



add to the C_3 rotations, another set of rotations by π about the fixed axes AA', B,B', CC'

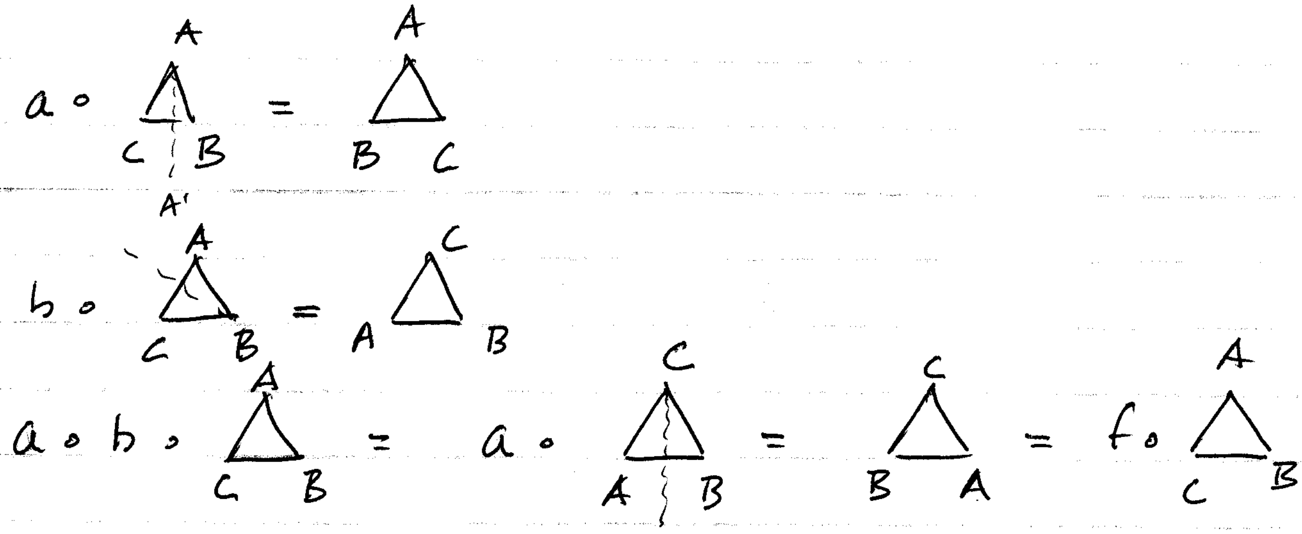
This describes the smallest of the "dihedral groups of n-fold axes", D_n:

1 n-fold axis set $\perp$ to plane of the figure ("principle axes")

n 2-fold axes $\perp$ to principle axes and symmetrically placed around it.

Here: D_3 which is isomorphic to S_3

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	f	d	c	b
b	b	d	e	f	a	c
c	c	f	d	e	b	a
d	d	b	c	a	f	e
f	f	c	a	b	e	d



etc.

So, you think group theory is about geometry?

Define 2 functions which take any symbol, x and transforms it to:

① $1-x$

② $\frac{1}{x}$

$x \xrightarrow{①} 1-x$

$1-x \xrightarrow{①} 1-(1-x) = x$ repeats

$1-x \xrightarrow{②} \frac{1}{1-x}$

$\frac{1}{1-x} \xrightarrow{①} 1-\left(\frac{1}{1-x}\right) = \frac{1-x-1}{1-x} = \frac{x}{x-1}$

$\frac{1}{1-x} \xrightarrow{②} 1-x$ etc.

$\frac{x}{x-1} \xrightarrow{①} 1-\left(\frac{x}{x-1}\right) = \frac{x-1-x}{x-1} = \frac{1}{1-x}$ etc.

$\frac{x}{x-1} \xrightarrow{②} \frac{x-1}{x}$

$\frac{x-1}{x} \xrightarrow{①} 1-\left(\frac{x-1}{x}\right) = \frac{x-x+1}{x} = \frac{1}{x}$ etc.

$\frac{x-1}{x} \xrightarrow{②} \frac{x}{x-1}$ etc.

These operations close on the same set of functions:

x	e
$1/x$	a
$1-x$	b
$x/x-1$	c
$1/1-x$	d
$x-1/x$	f