We have now finished the development of free quantum field theory. All non-interacting theories of interest in physics are describable in terms of the following Lagrange densities .. or their non-relativistic Hamiltonians

spin 0 : $\mathcal{L}(x) = (\partial_\mu\varphi)^+(\partial^\mu\varphi) - m^2\varphi^+\varphi$

spin 1/2 : $\mathcal{L}(x) = \bar{\psi}(x)\,(i\gamma^\mu\partial_\mu - m)\,\psi(x)$

spin 1 massless: $\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2$

spin 1 massive: $\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 B_\mu B^\mu$

By accepting the local gauge invariance idea as a primary requirement, we've even managed to develop our first interacting theory:

$$\mathcal{L}(x) = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x)}_{\text{called "kinetic terms"}} - e\,\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)$$

$$= \mathcal{L}_0(x) + \mathcal{L}_I(x)$$

The Hamiltonian density for this system is,

$$\mathcal{H}(x) = \bar{\pi}_j(x)\dot{\bar{\psi}}_j(x) + \pi_j(x)\dot{\psi}_j(x) + \pi_\mu \dot{A}^\mu - \mathcal{L}$$

12 150 $$\mathcal{H}(x) = \mathcal{H}_{\psi_0}(x) + \mathcal{H}_{A_0}(x) + \underbrace{e\,\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)}_{\mathcal{H}_I}$$

Problem: how do particles interact – evolve in time under the influence of the basic forces.–

Consider the evolution of a wavefunction from $t = t_0$ to t_1

$$|\xi(t_0)\rangle \longrightarrow |\xi(t_1)\rangle$$

some influence represented by an operator

$$T(t_1, t_0)\,|\xi(t_0)\rangle = |\xi(t_1)\rangle$$

assume that the norm doesn't change.

$$\langle \xi(t_1)|\xi(t_1)\rangle = \langle \xi(t_0)|T^{+}(t_1, t_0)\,T(t_1, t_0)|\xi(t_0)\rangle$$

$$= \langle \xi(t_1)|\xi(t_0)\rangle \quad \Rightarrow \quad T^{+}T = 1 \qquad \star$$

$$|\xi_2\rangle = T(t_2 t_1)|\xi_1\rangle$$

$$|\xi_1\rangle = T(t_1, t_0)|\xi_0\rangle$$

$$|\xi_2\rangle = T(t_2, t_1)\,T(t_1, t_0)|\xi_0\rangle = T(t_2, t_0)|\xi_0\rangle$$

(satisfies Group property)

let $t_2 = t_0$ $\qquad T(t_0 t_1)\,T(t_1 t_0) = T(\cancel{t_0} t_0) = 1$

$$\text{so} \quad T(t_1 t_0) = T^{-1}(t_0 t_1) \qquad \star\star$$

operate

$$\star \quad T^{+}(10)\,T(10) = 1$$

$$\star\star \quad T^{+}(10)\,T^{-1}(01) = 1 \quad \Rightarrow \quad T^{+}(10) = T(01) = T^{-1}(10)$$

a unitary transformation.

Consider a tiny time slice,

$$T(t_1, t_1 - \delta t)\, T(t_1 - \delta t, t_0) = T(t_1, t_0)$$

near the identity,

$$T(t, t-\delta t) = 1 - i\,\delta t\,\theta(t)$$

generator of time translation

$$T(t,t_0) = (1 - i\,\delta t\,\theta(t))\,T(t-\delta t, t_0)$$
$$= T(t-\delta t, t_0) - i\,\delta t\,\theta(t)\,T(t-\delta t, t_0)$$

$$\frac{T(t\,t_0) - T(t-\delta t, t_0)}{\delta t} = -i\theta(t)\,T(t-\delta t, t_0)$$

$$\lim_{\delta t \to 0} \quad \frac{\partial T(t\,t_0)}{\partial t} = -i\,\theta(t)\,T(t\,t_0)$$

(w/ initial conditions $T(t_0\,t_0) = 1$)

Can formally solve this,

$$T(t\,t_0) = 1 - i\int_{t_0}^{t} \theta(t')\,T(t'\,t_0)\,dt'$$ coming—

write above formal diff. eq.

$$\left[\frac{\partial}{\partial t} + i\,\theta(t)\right] T(t\,t_0) = 0$$

→ operate $|\xi_0\rangle$

$$\left[\qquad\right] T(t\,t_0)\,|\xi_0\rangle = 0$$

$$\frac{\partial}{\partial t}\,|\xi(t)\rangle = -i\,\theta(t)\,|\xi(t)\rangle$$

which we identify as the Schroedinger equation when

$$\theta(t) \equiv H = \frac{\hbar}{i}\frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial t}|\xi(t)\rangle = -i H |\xi(t)\rangle$$

This is the Schroedinger Picture.

Next we consider a state vector independent of time -- Heisenberg Picture state vectors -- which are connected to Sch. state vectors, where ~~operators~~ state vectors

H.P. S.P.

$$|\tilde{a}(t)\rangle = T^{-1}(t, t_0)\,|a(t)\rangle$$

and $\quad |a(t)\rangle = T(t, t_0)\,|a(t_0)\rangle \qquad$ S.P.

so

$$\begin{aligned}|\tilde{a}(t)\rangle &= T^{-1}(t\,t_0)\,T(t\,t_0)\,|a(t_0)\rangle \\ &= T(t_0\,t)\,T(t\,t_0)\,|a(t_0)\rangle \\ &= \underset{\text{(group property)}}{T(t_0\,t_0)\,|a(t_0)\rangle} = |a(t_0)\rangle = |\tilde{a}(t)\rangle\end{aligned}$$

constant

But operators!

$$\tilde{A}(t) = T^{-1}(t\,t_0)\,A\,T(t\,t_0) \qquad \checkmark$$

$$\frac{d\tilde{A}}{dt}(t) = \frac{\partial T^{-1}}{\partial t}(AT) + T^{-1}A\,\frac{\partial T}{\partial t}$$

adjoint of operator equation $\qquad i\dfrac{\partial T}{\partial t} = HT \qquad$ gives

$$-i\frac{\partial T^{+}}{\partial t} = T^{+}H$$

$$\frac{\partial T^{+}}{\partial t} = i\,T^{+}H$$

but $T^{+} = T^{-1}$
next page

$$\frac{d\tilde{A}(t)}{dt} = i T^{-1}HAT - i T^{-1}AHT$$
$$= i(T^{-1}HTT^{-1}AT - T^{-1}ATT^{-1}HT)$$

$$= i\,(\hat{H}\hat{A} - \hat{A}\hat{H})$$

$$\frac{d\tilde{\hat{A}}(t)}{dt} = i[\hat{H}, \hat{A}]$$

Now we have a whole new set of operators -- not state vectors any more. -

Free fields, say $\psi_j^0(x)$, the time dependence of which is governed by $\hat{H}_{\psi_0}$.

Remember, when we quantized we added harmonic time dependence which came from the equations of motion. .

$$a(t) = a(0)\,e^{-i\omega t}$$

for creation & annihilation operators.

In operator language, we must have instead

$$\hat{a}(t) = e^{i\hat{H}_{\psi_0}t}\,\hat{a}(0)\,e^{-i\hat{H}_{\psi_0}t}$$

$$= \hat{a}(0) + i\,[\underbrace{\hat{H}_{\psi_0}, \hat{a}(0)}_{= -\omega\hat{a}(0)}] + \dots$$

$$\hat{a}(t) = e^{-i\omega t}\,\hat{a}(0)$$ -- as we've been doing.

amplitudes are the same, up to phase

$$\langle a(t)|\hat{A}|a(t)\rangle_S = \langle\tilde{a}|\hat{\tilde{A}}|\tilde{a}\rangle_H$$

$$= \langle\tilde{a}|\hat{\tilde{A}}\,T^{-1}|a(t)\rangle$$

$$= \langle a(t)|T\hat{A}T^{-1}|a(t)\rangle$$

$$\hat{A} = T\hat{\tilde{A}}T^{-1}$$

$$T^{-1}\hat{A}T = \hat{\tilde{A}}$$

Now, now about our interacting ψ–γ model...

$$\psi_j(x,0) = \sum_{i=1}^{2} \int dk \left[\hat{a}^{(i)}(k,0)\, u_j^{(i)}(k)\, e^{i\vec{k}\cdot\vec{x}} + \hat{b}^{(i)\dagger}_{(k,0)}\, v_j^{(i)}(k)\, e^{-i\vec{k}\cdot\vec{x}} \right]$$

translate in time — complete Hamiltonian.

$$\hat{a}^{(i)}(k,t) = e^{i\hat{H}t}\, \hat{a}^{(i)}(k,0)\, e^{-i\hat{H}t}$$

This is a problem.

$$[\hat{H}_{A_0}, \hat{a}^{(i)}] = 0$$

$$[\hat{H}_{\psi_0}, \hat{a}^{(i)}] = -E a^{(i)}$$

but what's $[\hat{H}_I, \hat{a}^{(i)}] = ?$

time evolution may not necessarily be a phase change.

Notice that — equal times

$$\{a^i(k,t), a^{\dagger j}(k',t)\} = e^{iHt} a^j(k,0) e^{-iHt} e^{iHt} a^{\dagger j}(k,0) e^{-iHt} + e^{iHt} a^{\dagger j}(k,0) e^{-iHt} e^{iHt} a^j(k,0) e^{-iHt}$$

$$= e^{iHt} \{\hat{a}^i(k), \hat{a}^{\dagger j}(k)\} e^{-iHt}$$

$$= (2\pi)^3 2E\, \delta(\vec{k}-\vec{k}')\, \delta_{ij}\, e^{iHt} e^{-iHt}$$

$$= (2\pi)^3 2E\, \delta(\vec{k}-\vec{k}')$$

-- so equal time commutation relations hold, in presence of int.

The field operators themselves get their time dependence,

$$i[\hat{H}, \hat{\psi}_j(x)] = \frac{\partial \hat{\psi}_j(x)}{\partial t}$$

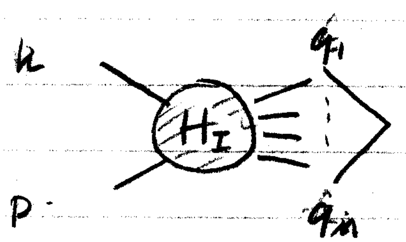
and $$i[\hat{H}, \hat{A}_\mu(x)] = \frac{\partial \hat{A}_\mu(x)}{\partial t}$$

full Hamiltonian → leading to different time evolution ⇒ many phases, not just one.

Scattering – in steps (← regular QM)

To keep track of this lets build up time dependent perturbation theory for scattering or decay.

Typically, we have



$$\langle \text{final states} | H_I | \text{initial states} \rangle$$

$|\text{initial}\rangle = |k\, p\rangle$ or $|P\rangle$

$|\text{final}\rangle = |q_1 q_2 \cdots q_n\rangle$

The amplitude is represented by

$$\mathcal{J}(n) = \langle \text{final} \,|\, H_I \,|\, \text{initial} \rangle$$

(excluding the channel in which nothing happens,
$|\text{final}\rangle = |\text{initial}\rangle$)

The probability is.

$$\text{average } \overline{\sum_{[i]}} \sum_{[f_n]} |\mathcal{J}(n)|^2$$

defined by beam and target

defined by the detector!

For a moment, define the volume for scattering -- over which the interaction is active.. V.

It's large and has N pairs of $|k\rangle$ and $|p\rangle$ in V at any time. We'll box normalize all states to this volume.

Probability per unit time to get a scatter from $[i] \to [f]$ is

$$\frac{\overline{\sum_{[i]}} \sum_{[f]} |\mathcal{J}(n)|^2}{T}$$

The cross section is

$d\sigma_{[f_n]}$ = effective area presented by V to incoming particles scattering into $[f_n]$

defined so that

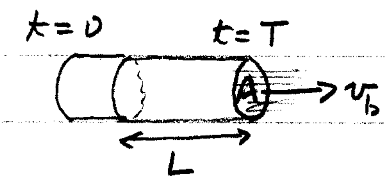
events recorded in $[f_n]$ = $d\sigma_{[f_n]}$ × relative flux per unit time

flux = # particles, k, passing by given target per unit area per unit time

relative flux = $\text{flux} \cdot N_t$

So, $$d\sigma_{[f_n]} = \frac{\left(|\mathcal{J}|^2/T\right)}{(\text{relative flux})} \times (\#\text{ final states})$$

rel flux: motivated by consideration of target frame viewpoint.



presume there is a beam density of $\rho = N_b/V$

$$\text{so flux} = \frac{N_b}{TA} = \frac{\rho V}{TA} = \frac{\rho \cancel{A} v_b \cancel{T}}{\cancel{T}\cancel{A}} = \frac{N_b}{V} v_b$$

$$\text{relative flux} = \frac{N_t N_b\,|\vec{v}_b - \vec{v}_t|}{V}$$ ← really, relative velocities

so, $$d\sigma \sim \int d^4x \frac{|\mathcal{T}|^2/T}{\text{relative flux}} \cdot \#\text{ final states}$$

1. Particle normalization $\int \psi^+ \psi \, dV = N$

so for each particle $\psi \sim \sqrt{N/V}$

we have: beam, target, n final state particles

so, $$|\mathcal{T}| \sim \left(\sqrt{\frac{N_f}{V}}\right)^n \left(\sqrt{\frac{N_t}{V}}\right)\left(\sqrt{\frac{N_b}{V}}\right)$$

2. density of final states

$$\left(\frac{V\int d^3q}{N_f}\right)^n$$

3. Integration over $\int d^4x \sim VT$

4. relative flux $\sim \frac{N_t N_b}{V}$

so,

$$d\sigma \sim \overset{(4)}{\left(\frac{\cancel{V}}{\cancel{N_t N_b}}\right)} \overset{(1)}{\left[\left(\sqrt{\frac{\cancel{N_f}}{\cancel{V}}}\right)^n \sqrt{\frac{\cancel{N_b}}{\cancel{V}}}\sqrt{\frac{\cancel{N_t}}{\cancel{V}}}\right]^2} \Big/ \cancel{T} \; \overset{(2)}{\left(\frac{\cancel{V}}{\cancel{N_f}}\right)^n} \; \overset{(3)}{\cancel{VT}}$$

$\longrightarrow$ no dependence on V, T, N's