Lecture 5 1/28/04 Lie Algebras

def. If it is possible to put a matrix representation into block-diagonal form, it is a Reducible Representation. If not, then it is an ~~IR~~ Irreducible Representation, IRR.

$$\Gamma^{(n)}(g_i) = \Gamma^{(m)}(g_i) \oplus \Gamma^{(l)}(g_i)$$

problem - similarity transformation

The connection with quantum mechanics comes from operators on functions — including basis functions in function spaces.

Define — $f(x,y) \Rightarrow f(|x\rangle) \rightarrow f'(|x\rangle) = \Gamma(g) f(|x\rangle)$
$= f[\,|x_i\rangle (\Gamma^{-1}(g))^i{}_j\,]$

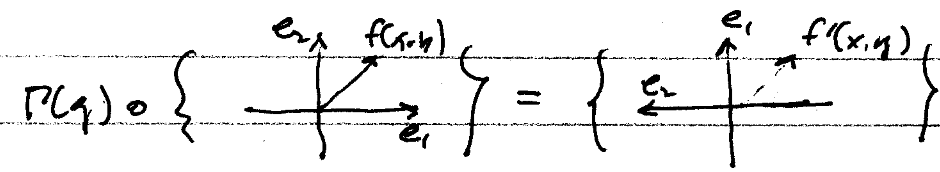
complicated... $f_i(|x\rangle)$

a vector space

another vector space.

Get different functions (f') when a transformation on coordinates ($|x\rangle$) is done

Suppose



$x \rightarrow -y$

$y \rightarrow +x$

So, if $f(x,y) = x + y$. Then transformation on $|x\rangle$

gives $f'(x,y) = -y + x$

new function of old coordinates

which ado inverse notation?

suppose $\Gamma(b)\, f(|x\rangle) = f[\,\Gamma^{-1}(b)|x\rangle] = g[|x\rangle]$

$$\Gamma(a)\Gamma(b)\, f(|x\rangle) = \Gamma(a)\, f[\,\Gamma^{-1}(b)\,|x\rangle]$$
$$= \Gamma(a)\, g\,[|x\rangle]$$
$$= g\,[\,\Gamma^{-1}(a)\,|x\rangle]$$

$$= f[\,\Gamma^{-1}(b)\,\Gamma^{-1}(a)\,|x\rangle]$$
$$= f\{[\Gamma(a)\,\Gamma(b)]^{-1}\,|x\rangle\}$$

$$= \Gamma(ab)\, f(|x\rangle)$$

so the operations in function space have the same multiplication table as the group elements themselves

For every set of operators, a space exists which is spanned by a set of functions, so that an operation one one $\to$ a linear combination of the others.

$$\phi_i \to \phi_i' = \Gamma(g)\,\phi_i\,(|x\rangle)$$
$$= \phi_j\,\Gamma(g)^j{}_i$$

analogues of $|e_j\rangle$

so, need $\Gamma^{-1}(g_i)|x\rangle$ for each group element

As usual $\quad |e_i\rangle \longrightarrow |e_i'\rangle = \Gamma |e_i\rangle$

vector: $\quad |x\rangle = |e_i\rangle x^i \quad$ original basis

$= |e_j'\rangle x^{j'} \quad$ transformed basis... new components.

$= \Gamma |e_j\rangle x^{j'} \quad$ new functions, original basis

So, components are related $\quad x^i = \Gamma^i{}_j x^{j'}$

or $\quad x' = \Gamma^{-1} x$

~~Gotta~~ Back to D_3...

$$\Gamma(a) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \left(= \Gamma^{-1}(a) \right)$$

need to find out $\quad \Gamma f(|x\rangle)$ by.

$$\Gamma^{-1}(a)|x\rangle = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} x^{1'} \\ x^{2'} \end{pmatrix} = \begin{pmatrix} -x^1 \\ x^2 \end{pmatrix}$$

So... functions. $\quad f(x,y) \longrightarrow \Gamma(a) f(x,y) = f'(\cancel{y, x})$ $-x, y$

eg. if $\quad f(x,y) = x^2 \longrightarrow f'(|x\rangle) = x^2$

PROBLEM for $\Gamma^{(2)}$ start out w/ $f(|x\rangle) = y \equiv f_1$

generate space under D_3

lots more ---

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LIE GROUPS

We could have characterized the finite groups this way:

$$\mathcal{G} = \{ g_1\, g_2\, g_3 \ldots \}$$

$$g_k = g_i \circ g_j$$

- fix j and vary i through the whole group.
- then, k runs through the whole group.

→ like dependent and independent variables.

So, the multiplication table could be thought of as a functional relationship

$$g_k = f(g_i, g_j)$$

Suppose the group element index is allowed to become continuous

$$g_i \rightarrow g(\alpha)$$

where one could define "continuous" in the usual way $g(a)$ and $g(\alpha')$ are 'near' if $|\alpha - \alpha'|$ is smaller than anything.

Such a group is the group of rotations, of which ~~B~~ C_3 is a subgroup.

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Same rules:

Suppose we have a single parameter α which can take on different values $\alpha', \alpha'', \alpha''' \ldots$

1. $g(\alpha) \circ g(\alpha') = g(\alpha'')$ "group property" — closure

2. $g(\alpha) \circ [g(\alpha') \circ g(\alpha'')] = [g(\alpha) \circ g(\alpha')] \circ g(\alpha'')$

3. $\exists_{\alpha^0} \ni \quad g(\alpha) \circ g(\alpha^0) = g(\alpha^0) \circ g(\alpha) = g(\alpha)$

4. $\forall_\alpha \ \exists \ \tilde{\alpha} \ni \quad g(\alpha) \circ g(\tilde{\alpha}) = g(\alpha^0)$

The group property says.

$$\alpha'' = f(\alpha, \alpha')$$

effectively, the multiplication table

This is a 1 parameter continuous group

In general there can be multi-parameter groups.

$$\alpha, \beta, \gamma, \delta, \epsilon \ldots \quad \text{or} \quad \alpha^1, \alpha^2, \alpha^3 \quad \text{or} \quad \alpha^\sigma \quad \text{or} \quad \vec{\alpha}$$

if $\sigma = r \Rightarrow$ r-parameter continuous group.

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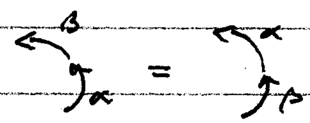
If. $\alpha^{\sigma''} = \varphi^\sigma(\vec{\alpha};\vec{\alpha}') = \varphi^\sigma(\alpha^1,\alpha^2,\dots\alpha^r;\alpha^{1\prime},\alpha^{2\prime},\dots\alpha^{r\prime})$

and φ^σ are analytic, differentiable in both arguments to all orders.. then the group is an "r-parameter Lie Group" after Sophus Lie (1842–1899) who did the essential work in collaboration with Felix Klein in the 1870's.

(There are little used (in practical terms) analogues to orthogonality and compactness...)

Consider the first continuous group → R_2, the group of rotations in 2 dimensions.

It's Abelian (~~...~~



parameter ranges $0 < \alpha < 2\pi$ and additive

$$\Gamma(\gamma) = \Gamma(\alpha)\circ\Gamma(\beta) \quad \Rightarrow \quad \gamma = \alpha+\beta$$

Because it's Abelian, its IRR are 1d.

Representation:

Need, function $\Gamma(\alpha) \ni \Gamma(\alpha+\beta) = \Gamma(\alpha)\Gamma'(\beta)$

and $\Gamma'(\alpha) = \Gamma(\alpha)\Gamma(0)$

The obvious solution: $\Gamma(\alpha) = e^{\alpha\Gamma'(0)}$

So that Γ be continuous and periodic $\Gamma(\alpha) = \Gamma(\alpha + 2\pi \cdot n)$

$\Rightarrow$ $\Gamma'(0)$ must be an imaginary integer.

Define $$m \equiv i\,\Gamma'(0) \qquad m = 0, \pm 1, \pm 2 \ldots$$

so

$$\Gamma(\alpha) = e^{-im\alpha}$$

It's naturally unitary and orthogonal

$$\int_0^{2\pi} \Gamma^{(m)+}(\alpha)\, \Gamma^{(m')}(\alpha)\, d\alpha = 2\pi\, \delta^{mm'}$$

The notion of class is -- elements are in the same class if they have the same value of the parameter.

Basis vectors? - can choose $|e_1\rangle$ and $|e_2\rangle$

So,

$$\left.\begin{aligned} R_3(\alpha)|e_1\rangle &= \cos\alpha\, |e_1\rangle + \sin\alpha\, |e_2\rangle \\ R_3(\alpha)|e_2\rangle &= -\sin\alpha\, |e_1\rangle + \cos\alpha\, |e_2\rangle \end{aligned}\right\} \text{mix.}$$

but ~~and~~ $R_3(\alpha)\left(|e_1\rangle \pm i\,|e_2\rangle\right) = e^{\mp i\alpha}\left(|e_1\rangle \pm i\,|e_2\rangle\right)$

So this means that

$|e_\pm\rangle \equiv |e_1\rangle \pm i\,|e_2\rangle$ transform according to the IRR of R_2 w/ $m = \pm 1$

Suppose we have a function of polar coordinates $\psi(r,\theta)$ in the plane,

$$\psi(r,\theta) \rightarrow \psi'(r,\theta') = \Gamma[R_3(\alpha)\psi(r,\theta)]$$

$$= \psi(r, R_3^{-1}\theta) = \psi(r, \theta-\alpha)$$

so,

$$\psi(r,\theta) = \psi(r)e^{im\theta} \quad \text{will transform}$$

according to the $\Gamma^{(m)}(\alpha)$ IRR of R_2.

We can expand an arbitrary function into components, each of which transforms according to a specific IRR of R_2:

$$\psi(r,\theta) = \sum_{m=-\infty}^{\infty} \psi_m(r)e^{im\theta}$$ of course, a Fourier expansion.

where

$$\psi_m(r) = \frac{1}{2\pi}\int_0^{2\pi} \psi(r,\alpha)e^{-im\alpha}\,d\alpha$$

Another group is related, The group of translations in 1d $\mathcal{T}_1$

$$x \rightarrow x+\alpha$$

same rule, same basis functions. and IRR.

$$\Gamma^{(k)}(\alpha) = e^{-ik\alpha} \quad \underline{\text{except}}$$

$0 < \alpha < \infty$ and k is not integer

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Still have same concerns: action of on functions, of say coordinates $\rightarrow$ function spaces and coordinate spaces.

Now... γ ~~depends~~ a group element depends on $\vec{\alpha} = \{\alpha^1, \alpha^2 \dots \alpha^r\} \Rightarrow \gamma_{\vec{\alpha}}$

Choose a basis $\quad \Gamma(\gamma_{\vec{\alpha}})\,|\eta_i\rangle\,\xi^i = |\eta_i\rangle\,\xi^{i\prime}$

$$|\eta_i\rangle\,\Gamma(\gamma_{\vec{\alpha}})^i{}_k\,\xi^k = |\eta_i\rangle\,\xi^{i\prime}$$

so

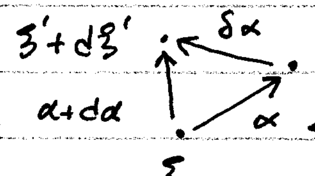
$$\xi^i \longrightarrow \xi^{i\prime} = \Gamma(\gamma_{\vec{\alpha}})^i{}_k\,\xi^k$$

$$= F^i(\xi^1\xi^2 \dots \xi^n; \alpha^1\alpha^2 \dots \alpha^r)$$

Differentiability is the key.

(Hammermesh) $\xi^{i\prime} = F^{i}(\vec{\xi}; \vec{\alpha})$

α takes points from $\xi \to \xi'$



But can also differentially go from $\xi \to \xi' + d\xi'$ through $\alpha + d\alpha \to$ because of the analyticity of ξ.

can get to $\xi' + d\xi'$ 2 ways. Can also get there from ξ' by going through $\delta\alpha$ which is very close to ϕ.

$$F(\xi; \alpha + d\alpha) = \xi' + d\xi'$$

and

$$F(\xi'; \delta\alpha) = \xi' + d\xi'$$

$$d\xi' = F(\xi'; \delta\alpha) - \xi'$$
$$= F(\xi'; \delta\alpha) - F(\xi'; 0)$$

or

$$d\xi' = \left[\frac{\partial F(\xi; \alpha)}{\partial \alpha}\right]_{\alpha=0} \delta\alpha \qquad \bigstar$$

for generally a set of ξ and α,

$$d\xi^{i} = \sum_{\sigma} \left[\frac{\partial F^{i}(\vec{\xi}; \vec{\alpha})}{\partial \alpha^{\sigma}}\right]_{\vec{\alpha}=0} \delta\alpha^{\sigma}$$

(coordinate label Latin, parameter Greek)

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call $\quad u^i_\sigma(\vec{\xi}) \equiv \left[\dfrac{\partial F^i(\vec{\xi};\vec{\alpha})}{\partial \alpha^\sigma}\right]_{\vec{\alpha}=0}$

so, $\quad d\xi^i = u^i_\sigma(\vec{\xi})\,\delta\alpha^\sigma$

Likewise for parameters -- related through the group property

$$g(\alpha) = g(\alpha')\circ g(\alpha'')$$

$$\alpha = \varphi(\alpha^{1\prime} \ldots \alpha^{r\prime}; \alpha^{1\prime\prime} \ldots \alpha^{r\prime\prime})$$

let $\quad \vec{\alpha} = \{\vec{\alpha}\} \qquad \vec{\beta} = \{\vec{\alpha}'\} \qquad \vec{\epsilon} = \{\vec{\alpha}''\}$

$$g(\vec{\alpha}) = g(\vec{\beta})\circ g(\vec{\epsilon})$$

or. $\quad \alpha^\sigma = \varphi^\sigma(\vec{\beta};\vec{\epsilon})$

which says $\quad \alpha^\sigma + d\alpha^\sigma = \varphi^\sigma(\alpha^1 \ldots \alpha^r; \delta\alpha^1 \ldots \delta\alpha^r)$

The differentiability of Lie groups ⇒ can be treated as an analytic function of r variables.

$$d\alpha^\sigma = \underbrace{\sum_{\rho=1}^{r}\left[\frac{\partial\varphi^\sigma(\vec{\alpha};\vec{\beta})}{\partial\beta^\rho}\right]_{\vec{\beta}=0}}_{\text{III}} \delta\alpha^\rho$$

$$= V^\sigma_\rho(\vec{\alpha})\,\delta\alpha^\rho \qquad\qquad V^\sigma_\rho(0) = \delta^\sigma_\rho$$

The inverse exists $\qquad V^\sigma{}_\rho \Lambda^\rho{}_\tau = \delta^\sigma{}_\tau$

so

$$\Lambda^\rho{}_\tau \, d\alpha^\tau = \Lambda^\rho{}_\tau V^\sigma{}_\rho \, \delta\alpha^\rho = \underline{\underline{\delta^\sigma{}_\tau}} \, \delta\alpha^\rho$$

$$\Lambda^\rho{}_\tau \, d\alpha^\tau = \delta\alpha^\rho$$

Then vector change is

$$d\xi^i = u^i{}_\sigma(\vec{\xi}) \Lambda^\sigma{}_\rho(\vec{\alpha}) \, d\alpha^\rho$$

or $$\frac{d\xi^i}{d\alpha^\rho} = u^i{}_\sigma(\vec{\xi}) \Lambda^\sigma{}_\rho(\vec{\alpha}) \qquad (1)$$

the rate of change of ξ wrt α from $\xi = 0$ when $\alpha = 0$

The effect of groups on function

$$\Gamma(\gamma_\mu) f(\vec{\xi}) = f'(\xi) = f(\vec{\xi}\,')$$

since $$f(\vec{\xi}\,') = f(\vec{\xi} + \delta\vec{\xi}) \xrightarrow[\text{limit}]{} \frac{\partial f(\vec{\xi})}{\partial \xi^i} d\xi^i + f(\vec{\xi})$$

so

$$f(\vec{\xi}\,') - \xi(\vec{\xi}) = \Delta f(\vec{\xi}) = \frac{\partial f(\vec{\xi})}{\partial \xi^i} d\xi^i = df(\vec{\xi}) \qquad (2)$$

(1) & (2) and $\delta\xi$ is as $\alpha \to 0$

$$df(\vec{\xi}) = \frac{\partial f(\vec{\xi})}{\partial \xi^i} u^i{}_\sigma(\vec{\xi}) \, \delta\alpha^\sigma$$

rearrange

$$df = \delta\alpha^\sigma\, u^i{}_\sigma(\vec{\xi})\, \frac{\partial}{\partial \xi^i} f(\xi)$$

$$\equiv \delta\alpha^\sigma X_\sigma f(\vec{\xi}) \quad \Rightarrow \quad \underline{\underline{f(\vec{\xi}') = f(\vec{\xi}) + \delta\alpha^\sigma X_\sigma f(\vec{\xi})}}$$

defining $X_\sigma \equiv u^i{}_\sigma \frac{\partial}{\partial \xi^i}$ as

the <u>Infinitesimal Generators</u> — where

$u^i{}_\sigma$ describes how coordinates infinitesimally change wrt infinitesimal change in parameters.

For $f = \xi^i$

$$\Gamma(\gamma)\xi^i = \xi^{i\prime} = \xi^i + d\xi^i$$

$$= \xi^i + \delta\alpha^\sigma X_\sigma \xi^i$$

$$= \xi^i + u^j{}_\sigma \frac{\partial \xi^i}{\partial \xi^j} \delta\alpha^\sigma$$

$$= \xi^i + u^i{}_\sigma \delta\alpha^\sigma$$ which is where we started

So. Let's imagine a functional basis which transforms under this group.

The operator $1 + \delta x^\sigma X_\sigma$ is close to ~~unity~~ the identity.

So, the group operation on an arbitrary function:

$$\Gamma(\gamma_{\vec{\alpha}}) f(\vec{\xi}) = f(\vec{\xi}) + \delta x^\sigma X_\sigma f(\vec{\xi})$$
$$= (1 + \delta x^\sigma X_\sigma) f(\vec{\xi})$$

Consider 2 successive operations.

$$\Gamma(\gamma_{\vec{\alpha}})\Gamma(\gamma_{\vec{\beta}}) = (1 + \delta\alpha^\sigma X_\sigma)(1 + \delta\beta^\rho X_\rho)$$
$$= 1 + \delta\beta^\rho X_\rho + \delta\alpha^\sigma X_\sigma + \mathcal{O}(\delta\alpha\,\delta\beta)$$
$$= 1 + \delta\beta^1 X_1 + \delta\beta^2 X_2 + \cdots + \delta\alpha^1 X_1 + \delta\alpha^2 X_2 + \cdots$$

$$= 1 + \underbrace{(\delta\beta^\sigma + \delta x^\sigma)}_{\delta\epsilon^\sigma} X_\sigma$$

$\delta\epsilon^\sigma \longrightarrow$ additive property of parameters

$$\Gamma(\gamma_{\vec{\epsilon}}) = 1 + \delta\epsilon^\sigma X_\sigma = \Gamma(\gamma_{\vec{\alpha}})\Gamma(\gamma_{\vec{\beta}})$$

group property.

Let α be only a small $\delta\beta$

$$\Gamma(\vec{\alpha})\Gamma(\vec{\beta}) = \Gamma(\vec{\alpha}+\vec{\beta})$$
$$\Gamma(\delta\vec{\beta})\Gamma(\beta) = \Gamma(\delta\vec{\beta}+\vec{\beta})$$
$$(1+\delta\beta^\sigma X_\sigma)\Gamma(\vec{\beta}) = \Gamma(\vec{\beta}) + \delta\beta^\sigma X_\sigma \Gamma(\vec{\beta}) = \Gamma(\vec{\beta}+\delta\vec{\beta})$$

$$\frac{\Gamma(\vec{\beta}+\delta\vec{\beta}) - \Gamma(\vec{\beta})}{\delta\beta^\sigma} = X_\sigma \Gamma(\vec{\beta})$$

$$\lim_{\delta\beta^\sigma\to 0} \qquad \frac{d\Gamma(\vec{\beta})}{d\beta^\sigma} = X_\sigma \Gamma(\vec{\beta})$$

$$\frac{d\Gamma}{\Gamma} = X_\sigma d\beta^\sigma$$

which is an operator equation that can be formally solved...

$$\Gamma(\vec{\beta}) = e^{X_\sigma \beta^\sigma}$$

Imagine dividing the parameters into small pieces,

$$\vec{\beta} = \vec{\theta}/n \qquad \Gamma(\vec{\beta}) = e^{X_\sigma \theta^\sigma/n}$$

and let n be very large

$$\Gamma(\theta/n) = \lim_{n\to\infty} e^{X_\sigma\theta^\sigma/n} = 1 + X_\sigma\theta^\sigma/n$$

Putting many of them together

$$\Gamma(\theta/n)\Gamma(\theta/n)\ldots = (1+X_\sigma\theta^\sigma/n)^n = e^{X_\sigma\theta^\sigma}$$

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so, the finite transformation θ can be thought of as a build-up of infinitesimal transformations θ/n where $n \to \infty$

Go back to $$\frac{\partial \xi^i}{\partial \alpha^\rho} = u^i_\sigma(\vec{\xi})\Lambda^\sigma_\rho(\vec{\alpha})$$

Demand $$\frac{\partial^2 \xi^i}{\partial\alpha^\tau\partial\alpha^\rho} = \frac{\partial^2 \xi^i}{\partial\alpha^\rho\partial\alpha^\tau}$$ $\Rightarrow$ arbitrariness wrt α.

$$u^i_\sigma(\vec{\xi})\frac{\partial\Lambda^\sigma_\rho(\vec{\alpha})}{\partial\alpha^\tau} + \frac{\partial u^i_\sigma(\vec{\xi})}{\partial\alpha^\tau}\Lambda^\sigma_\rho(\vec{\alpha}) = u^i_\sigma(\vec{\xi})\frac{\partial\Lambda^\sigma_\tau(\vec{\alpha})}{\partial\alpha^\rho} + \frac{\partial u^i_\sigma(\vec{\xi})}{\partial x^\rho}\Lambda^\sigma_\tau(\vec{\alpha})$$

$$\frac{\partial u^i_\sigma(\vec{\xi})}{\partial\xi^j}\underbrace{\frac{\partial\xi^j}{\partial\alpha^\tau}}_{u^j_\nu\Lambda^\nu_\tau}\Lambda^\sigma_\rho(\vec{\alpha})$$

$$\frac{\partial u^i_\sigma(\vec{\xi})}{\partial\xi^j}\underbrace{\frac{\partial\xi^j}{\partial\alpha^\rho}}_{u^j_\epsilon\Lambda^\epsilon_\rho}\Lambda^\sigma_\tau(\vec{\alpha})$$

collect and rearrange terms:

$$u^j_\nu\frac{\partial u^i_\sigma}{\partial\xi^j}\Lambda^\nu_\tau\Lambda^\sigma_\rho - u^j_\epsilon\frac{\partial u^i_\sigma}{\partial\xi^j}\Lambda^\epsilon_\rho\Lambda^\sigma_\tau = u^i_\sigma\left(-\frac{\partial\Lambda^\sigma_\rho}{\partial\alpha^\tau} + \frac{\partial\Lambda^\sigma_\tau}{\partial\alpha^\rho}\right)$$

multiply by. $V^\tau_\rho V^\rho_\eta$

$$\delta^\sigma_\rho \qquad \delta^\epsilon_\eta$$

$$u^j_\nu \frac{\partial u^i_\sigma}{\partial \xi^j} V^\tau_\rho V^\rho_\eta \Lambda^\nu_\tau \Lambda^\sigma_\rho - u^j_\epsilon \frac{\partial u^i_\sigma}{\partial \xi^j} V^\tau_\rho V^\rho_\eta \Lambda^\epsilon_\rho \Lambda^\sigma_\tau$$

$$\delta^\nu_\rho \qquad \delta^\sigma_\eta$$

$$= (\quad) u^i_\sigma V^\tau_\rho V^\rho_\eta$$

$$u^j_\rho \frac{\partial u^i_\eta}{\partial \xi^j} - u^j_\eta \frac{\partial u^i_\rho}{\partial \xi^j} = \left(\frac{\partial \Lambda^\sigma_\tau}{\partial \alpha^\rho} - \frac{\partial \Lambda^\sigma_\rho}{\partial \alpha^\tau} \right) V^\tau_\rho V^\rho_\eta u^i_\sigma$$

$$\equiv c^\sigma_{\rho\eta} u^i_\sigma$$

operate by $\frac{\partial}{\partial \xi^i}$ from the right.

$$u^j_\rho \frac{\partial}{\partial \xi^j} u^i_\eta \frac{\partial}{\partial \xi^i} - u^j_\eta \frac{\partial}{\partial \xi^j} u^i_\rho \frac{\partial}{\partial \xi^i} = c^\sigma_{\rho\eta} u^i_\sigma \frac{\partial}{\partial \xi^i}$$

recognize the infinitesimal generator $X_\sigma = u^k_\alpha \frac{\partial}{\partial \xi^k}$
and --- ta dah!

$$X_\rho X_\eta - X_\eta X_\rho = c^\sigma_{\rho\eta} X_\sigma$$

$$[X_\rho, X_\eta] = c^\sigma_{\rho\eta} X_\sigma \qquad \text{Lie Algebra}$$

called "Structure Constants"

Fundamental Theorem of Lie: everything you need to know about a Lie Group is in the algebra satisfied by the generators

⇒ structure constants completely determine the group

⇒ don't need the multiplication table

consider $\mathcal{T}_1$ -- the 1 dimensional shift

$$\xi \xrightarrow{\mathcal{T}_1} \xi' = a\xi + b$$

steps... need parameter value of zero to represent no change

$$\xi \to \xi' = (1+\alpha)\xi + \alpha_2$$

so, the infinitesimal transformation is

$$\xi' = \xi + d\xi = (1 + \delta\alpha_1)\xi + \delta\alpha_2$$

$$= \xi + \delta\alpha_1 \xi + \delta\alpha_2$$

$$\delta\xi = \xi' - \xi = \delta\alpha_1 \xi + \delta\alpha_2$$

remember that $\delta\xi^i = u^i{}_\sigma \delta\alpha^\sigma$

or

$$\xi' = u'{}_\sigma \delta\alpha^\sigma$$

only have $i=1$ for 1d

$$= u'{}_1 \delta\alpha' + u'{}_2 \delta\alpha^2$$

so, we can recognize

$$u'{}_1 = \xi \quad \text{and} \quad u'{}_2 = 1$$

and can construct the generators

$$X_\sigma = u^i{}_\sigma \frac{\partial}{\partial\xi^i} \Rightarrow \quad X_1 = u'{}_1 \frac{\partial}{\partial\xi'} = \xi' \frac{\partial}{\partial\xi'}$$

$$X_2 = u'{}_2 \frac{\partial}{\partial\xi'} = \frac{\partial}{\partial\xi'}$$

so, need to operate on something

$$[X_1, X_2] F = \xi \frac{\partial}{\partial \xi^1} \frac{\partial}{\partial \xi^1} F - \frac{\partial}{\partial \xi^1} \xi^1 \frac{\partial}{\partial \xi^1} F$$ drop ξ^1's

$$= \xi \frac{\partial^2 F}{\partial \xi^2} - \frac{\partial F}{\partial \xi} - \xi \frac{\partial^2 F}{\partial \xi^2}$$

$$= -\frac{\partial F}{\partial \xi} = -X_2 F$$

so: $[X_1, X_2] = -X_2$ is the Lie Algebra for $\mathcal{T}_1$

and the structure constants are

$$c^1_{12} = 0 = c^1_{21}$$

$$c^2_{12} = -1 = -c^2_{21}$$