Vector Spaces. → almost unchanged from Grassmann, mid 1800's

def: scalar – any abstract entity satisfying primary combinatoric relations

I. To every pair a and b of scalars, there corresponds another scalar $a \oplus b$ → "sum" or "addition" of a and b.

A. closure

B. commutative $a \oplus b = b \oplus a$

C. associative $a \oplus (b \oplus c) = (a \oplus b) \oplus c$

D. zero: $\exists\ \phi$ such that $a \oplus \phi = a$ → unique

E. $\exists\ d \ni \forall_a$ $a \oplus d = \phi$, $d = -a$ → <u>negative</u>

II. To every pair a and b, $\exists$ another scalar $a \otimes b$ which is the <u>product</u> or <u>multiplication</u> of a & b

A. closure

B. commutative $a \otimes b = b \otimes a$ (Hamilton)

C. associative $a \otimes (b \otimes c) = (a \otimes b) \otimes c$

D. $\exists\ \underline{\mathbb{1}} \ni \forall_a$ $a \otimes \underline{\mathbb{1}} = a$ <u>identity</u>

E. $\exists\ c \ni \forall_c$ $a \otimes c = \mathbb{1}$ $c = a^{-1}$, <u>inverse</u>

F. $a \otimes (b \oplus c) = a \otimes b \oplus b \otimes c$ dist. wrt addition

def: a field is any set of entities which satisfies I and II.

→ the general scalar field $\mathcal{F}$.

examples: set of real numbers $\mathbb{R}$

complex numbers $\mathbb{C}$

integers $\mathbb{Z}$

def: a vector space is a set $\mathcal{V}$ of elements which satisfy the conditions:

III. To every pair ξ and $\eta \in \mathcal{V}$, there is another vector $\xi \oplus \eta \in \mathcal{V}$ called the sum.

NOTATION: lower case Latin: scalars

lower case Greek: vectors

also Dirac Notation: $\xi \longleftrightarrow |\xi\rangle$

A. $\xi \oplus \eta = \eta \oplus \xi$ $\qquad |\xi\rangle \oplus |\eta\rangle = |\eta\rangle \oplus |\xi\rangle$

B. $\xi \oplus (\eta \oplus \gamma) = (\xi \oplus \eta) \oplus \gamma$

C. $\exists\ \phi \ni \xi \oplus \phi = \xi$ origin

D. $\forall_\xi\ \exists\ \eta \in \mathcal{V} \ni \xi \oplus \eta = 0,\ \eta = -\xi$, inverse

IV. To every pair $a \in \mathcal{F}$ and $\xi \in \mathcal{V}$, there is a vector $a \cdot \xi \in \mathcal{V}$

A. $a \otimes (b \cdot \xi) = (a \otimes b) \cdot \xi$

$a \otimes (b|\xi\rangle) = a \otimes b\,|\xi\rangle = |\xi\rangle(a \otimes b)$

B. $\mathbb{1} \cdot \xi = \xi$

C. $a \otimes (\xi \oplus \eta) = a\xi \oplus a\eta$

$a \cdot (|\xi\rangle \oplus |\eta\rangle) = |\xi\rangle a \oplus |\eta\rangle b$

D. $(a \oplus b) \cdot \xi = a \cdot \xi \oplus b \cdot \xi$

def: A space is said to be spanned by a set of vectors if every vector in that space can be represented by a linear combination

III & IV specify a Linear Vector Space, L

V. $\mathcal{V}$ is <u>unitary</u> if there is an additional operation called the scalar product. (ξ, η) which is scalar-valued (real or complex).

A. $(\xi, \eta) = (\eta, \xi)^* = a$

B. $(\xi, \eta \oplus \gamma) = (\xi, \eta) \oplus (\xi, \gamma)$

C. $(\xi, a\eta) = a(\xi, \eta)$

D. 1. $(\xi, \xi) \geq \phi$

2. $(\xi, \xi) = \phi$ iff $\xi = \phi$

E. 1. $(\xi, a\eta \oplus b\gamma) = a(\xi, \eta) \oplus b(\xi, \gamma)$ linear.

2. $(a\xi \oplus b\eta, \gamma) = a^*(\xi, \gamma) \oplus b^*(\eta, \gamma)$ antilinear

VI. <u>Outer product</u> $[\xi\eta] = -[\eta\xi]$ is non-commutative multiplication

Strictly speaking, Unitary Vector Spaces are Vector Spaces "endowed with a metric"

Having both linear and antilinear scalar products $\{(\xi, \eta)$ is <u>linear</u> in η and <u>antilinear</u> in ξ.$\}$ suggests 2 different vector spaces containing, say, ξ and η which are linear in ~~them~~ themselves and antilinear wrt one another.

$\longrightarrow$ spaces are <u>dual</u> to one another. $\Rightarrow$ Dirac's notation is nice

$$|\eta\rangle \in \mathcal{V} \quad \text{and} \quad \langle\xi| \in \tilde{\mathcal{V}}$$

then

IV A. $\langle \xi | \eta \rangle = \langle \eta | \xi \rangle^{\times}$

IV B $\langle \xi | \eta \oplus \gamma \rangle = \langle \xi | \eta \rangle \oplus \langle \xi | \gamma \rangle$

IV E1 $\langle \xi | a\eta \oplus b\gamma \rangle = \langle \xi | \eta \rangle a \oplus \langle \xi | \eta \rangle b$

IV E2 $\langle a\xi \oplus b\eta | \gamma \rangle = \langle \xi | \gamma \rangle a^{*} \oplus \langle \eta | \gamma \rangle b^{*}$

All abstract $\longrightarrow$ Representations make it concrete.. then vectors are the "Representations".

Typically... an ordered list of complex scalars

$\xi = \{ \xi^1, \xi^2, \xi^3 \ldots \xi^n \}$ the "components"

a contravariant vector.

so, can recast some of above

III A. $\xi \oplus \eta = \gamma$

$\{ \xi^1 \oplus \eta^1, \xi^2 \oplus \eta^2, \ldots \} = \{ \gamma^1, \gamma^2, \ldots \}$

components: $\gamma^1 = \xi^1 \oplus \eta^1$

IV. $a \cdot \xi = \{ a\xi^1, a\xi^2, \ldots \}$

IV D1. $\langle \xi | \xi \rangle \equiv |\xi|^2 = |\xi^1|^2 \oplus |\xi^2|^2 \oplus |\xi^3|^2 \oplus \ldots$

where

$|\xi^i|^2 \equiv \xi^{i\times} \xi^i$

geometry leaps out at you —. Grassmann

Define $|\xi| \equiv +\sqrt{\langle \xi | \xi \rangle}$ as the "length" of ξ

if $|\xi| = \mathbb{1}$, then ξ is normalized

Scalar product now:

$$\langle \xi | \eta \rangle = \sum_{i=1}^{n} \xi_i^{+} \eta^i$$

where define $\xi_i^{+} \equiv \xi^{i\times}$

Einstein summation: $\sum_{i=1}^{n} \underbrace{\xi_i^{+} \eta^i}_{\text{upper and lower.}} \equiv \xi_i^{+} \eta^i$

Schwartz Inequality
$$|\langle \xi | \eta \rangle| \leq |\xi| \, |\eta|$$
$$|\langle \xi | \eta \rangle| \leq \sqrt{\langle \xi | \xi \rangle} \sqrt{\langle \eta | \eta \rangle}$$

If $\langle \xi | \eta \rangle = 0$, then ξ and η are orthogonal.

Linear Combination: Let $\alpha, \beta, \gamma \ldots \in \mathcal{V}$ and $a_i \in \mathcal{C}$

$$\alpha_0 = a_1 \alpha \oplus a_2 \beta \oplus \cdots \oplus a_d \xi$$

are a linear combination -- a_i's are coefficients

If $a_i = 0 \;\; \forall_i \Rightarrow \alpha_0 = 0$.

But $\alpha_0 = 0$ w/out all $a_i = 0 \rightarrow$ linear dependence

$$\beta = \frac{a_1}{a_2} \alpha \oplus \frac{a_3}{a_2} \gamma \oplus \cdots$$

<u>Linear Independence</u>: generally, n variables x_i are L.I. if

$$\sum_{i=1}^{n} c_i x_i = 0$$ is <u>not</u> <u>true</u> for values of c_i other than $c_i = 0$

ie. none of the x_i can be written in terms of others.

NOTATION :

a) $\vec{\phi}_j$

index labels different vectors: ie $\vec{\phi}_1 = \vec{\alpha}$, $\vec{\phi}_2 = \vec{\beta}$ } components hidden.

or w/out "$\rightarrow$" $\phi_j \rightarrow \phi_{ji}$ to label components.

b) $|\phi^j_{(i)}\rangle$ (no $\rightarrow$)

j^{th} component of i^{th} vector.

$$|\phi_{(i)}\rangle = \{ \phi^1_{(i)}, \phi^2_{(i)}, \ldots \phi^d_{(i)} \}$$

linear independence: $|\phi_{(i)}\rangle a^i = 0$

$\Rightarrow \forall_i \; a^i = 0$

a) For d-dimensional V, there are no more than d L.I. vectors spanning V

b) The vectors belonging to a set of mutually orthogonal vectors form a linearly independent set.

Familiar set, $\mathcal{E}_3$.

Summarize handout:

Characterized

Field: combination rules for scalars under

I. "addition" $\oplus$ eg $a \oplus b$

II. "multiplication" $\otimes$ + distributive property btw $\oplus$ & $\otimes$

examples: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$

Vectors and vector space $\mathcal{V}$: combination rules for vectors under:

III. "addition" $\oplus$ eg. $\xi \oplus \eta$ or $|\xi\rangle \oplus |\eta\rangle$

IV. "scalar multiplication" eg $a \cdot \xi \in \mathcal{V}$ or $|\xi\rangle a \in \mathcal{V}$

def. A space $\mathcal{V}$ is spanned by a set S of vectors if every vector $\in \mathcal{V}$ can be rep. by a linear combination of vectors $\in S$.

III + IV specify a Linear Vector Space, L.

V. $\mathcal{V}$ is Unitary if there is the operation of scalar or inner product eg. (ξ, η) or $\langle \xi | \eta \rangle$ which is scalar-valued.

important property:

~~linearity~~: $\cancel{(a\xi \oplus b\eta, \gamma) = a^*(\xi,\gamma) \oplus b^*(\eta,\gamma)}$

antilinearity: $(a\xi \oplus b\eta, \gamma) = a^*(\xi,\gamma) \oplus b^*(\eta,\gamma)$

linearity: $(\xi, a\eta \oplus b\gamma) = a(\xi,\eta) \oplus (\xi,\gamma)$

30b

$\Rightarrow$ existence of 2 spaces

$$|\eta\rangle \in \mathcal{V} \quad \text{and} \quad \langle \xi| \in \hat{\mathcal{V}}$$

dual to $\mathcal{V}$

η and ξ are linear wrt themselves

antilinear wrt one another.

VI. "outer product" $[\xi, \eta] = -[\eta, \xi]$ N.C.

Representations

VI. Consider a set of vectors $\eta_{(i)} \in \mathcal{V}$: $|\eta_1\rangle, |\eta_2\rangle, |\eta_3\rangle \cdots |\eta_n\rangle$

each a full-fledged vector $\in \mathcal{V}$

Such a set is a <u>BASIS</u> in $\mathcal{V}$ if

1. They are linearly independent:

$$\sum_{i=1}^{n} |\eta_{(i)}\rangle a^i = 0 \quad \text{iff} \quad \forall_i \quad a^i = 0.$$

<u>Then</u>

2. Every vector, $\xi \in \mathcal{V}$, can be written as a linear combination of $|\eta_{(i)}\rangle$:

$$|\xi\rangle = \sum_{i=1}^{n} |\eta_{(i)}\rangle c^i$$

scalars

~~if the ... for 2 basis sets ...~~ $\cancel{\langle \eta | \phi \rangle = \sum_{i=1}^{n} \eta}$

30c

The norm: $\langle \eta_{(i)} | \eta_{(j)} \rangle = \delta_{ij} = 0 \quad i \neq j$

$= 1 \quad i = j$

$\Rightarrow \eta_{(i)}$ are orthonormal representatives.

So, determine c^i's.

$$\langle \eta_{(j)} | \xi \rangle = \sum_{i=1}^{n} \langle \eta_{(j)} | \eta_{(i)} \rangle c^i$$

$$\langle \eta_{(j)} | \xi \rangle = \sum_{i=1}^{n} \delta_{ji} c^i = c_j$$

or $\langle \eta^{(j)} | \xi \rangle = c^j$

then $$| \xi \rangle = \sum_{i=1}^{n} | \eta_{(i)} \rangle \underbrace{\langle \eta^{(i)} | \xi \rangle}_{\text{geometrical concept of projection.}}$$

Define. $\xi^i \equiv \langle \eta^{(i)} | \xi \rangle = c^i$ and then,

$$| \xi \rangle = \sum_{i=1}^{n} | \eta_{(i)} \rangle \xi^i$$

So, in this particular basis

$$| \xi \rangle_\eta = \underbrace{\{ \xi^1, \xi^2, \ldots \xi^n \}}_{\text{components}}$$

The length, or norm, of ξ is

$$\langle \xi | \xi \rangle \equiv |\xi|^2 = |\xi^1|^2 \oplus |\xi^2|^2 \oplus \cdots \oplus |\xi^n|^2$$

where generally $|\xi^i|^2 \equiv \xi^{i*} \xi^i$

related to the dual space.

We can recover a completely self-consistent tensor representation by defining

$$\xi^+_i \equiv \xi^{i*}$$

So,

$$\langle \xi | \rho \rangle_\eta = \sum_{i=1}^{n} \xi^{i*} \rho^i = \sum_{i=1}^{n} \underbrace{\xi^+_i \rho^i}_{\text{up-down}} = \xi^+_i \rho^i$$

up-down $\longrightarrow$ Einstein summation convention.

With this the earlier discussion of O.N. bases becomes

$$\langle \eta^{(j)} | \eta_{(i)} \rangle = \delta^j_i$$

convenient set of L.I. vectors $\in \mathcal{E}_3$:

$$\hat{i} = \{1,0,0\} \quad \hat{j} = \{0,1,0\} \quad \hat{k} = \{0,0,1\}$$

mutually $\perp$

$$\langle i|j\rangle = \langle i|k\rangle = \langle j|k\rangle = 0$$

$\exists$ no $a_i \ni \quad a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = 0$

Usefull notation: $|\hat{e}_{(i)}\rangle$: $|\hat{e}_{(1)}\rangle = \hat{i} = \{1,0,0\}$

$$|\hat{e}_{(2)}\rangle = \hat{j}$$
$$|\hat{e}_{(3)}\rangle = \hat{k}$$

so, $|\hat{e}_{(1)}\rangle = \{ e^1_{(1)}, e^2_{(1)}, e^3_{(1)} \}$
$= \{1,0,0\}$

Orthogonality and normalization:

$$\langle \hat{e}^{(j)} | \hat{e}_{(i)} \rangle = \delta^j{}_i \qquad \delta^j{}_i = 0 \quad i \neq j$$
$$= 1 \quad i = j$$

dual has index <u>up</u>

~~Suppose we have a set of d L.I., O.N. vectors $\in \mathcal{V}$ which we try to expand by one more:~~

~~$$\langle \eta^{(j)} | \eta_{(i)} \rangle = \delta^j{}_i \qquad \text{O.N.}$$~~

~~ie $$\sum_{i=1}^{d} |\eta_{(i)}\rangle a^i \oplus |\xi\rangle b = 0$$~~

~~$$|\xi\rangle = -|\eta_{(i)}\rangle \frac{a^i}{b}, \quad c^i \equiv a^i/b$$~~

an abstract object $\longrightarrow$ ~~$|\xi\rangle = |\eta_{(i)}\rangle c^i$~~ $\longleftarrow$ expansion coef. for $\vec{\xi}$

Basis Vectors spanning $\mathcal{V}$

drop the (): $|e_{(i)}\rangle \to |e_i\rangle$

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Bases are not unique – suppose another set. $\in V$

$$|\xi\rangle = |\eta_i\rangle \xi^i \text{ \& }$$
$$|\xi\rangle = |e_k\rangle d^k = \underline{\underline{|e_k\rangle \xi^{k'}}}$$

project $\quad \langle e^m|\xi\rangle = \langle e^m|e_k\rangle \xi^{k'} = \delta^m{}_k \xi^{k'} = \xi^{m'}$

so as before

$$|\xi\rangle = |e_k\rangle\langle e^k|\xi\rangle$$

Same vector, 2 different representations –

$$|\xi\rangle = |e_k\rangle \xi^{k'} = |\eta_i\rangle\langle \eta^i|\xi\rangle \qquad ①$$
$$② \qquad = |\eta_i\rangle\langle\eta^i|e_\ell\rangle \xi^{\ell'}$$

expand one basis in other $\qquad |\eta_{(i)}\rangle = |e_{(n)}\rangle a^n_{(i)}$
$$= |e_{(n)}\rangle \eta^n_{(i)} \qquad ③$$
$$\text{\& } \langle \eta^{(i)}| = \langle e^{(n)}|\eta^{(i)}{}_n \qquad ④$$

Then ① $\qquad |e_{(k)}\rangle \xi^{k'} = ~~|e_\ell~~$

scalar

$$= \left(|\eta_{(i)}\rangle\langle e^{(n)}|\eta^{(i)}{}_n\right)\left(|e_{(\ell)}\rangle \xi^{\ell'}\right)$$
$$= |\eta_{(i)}\rangle\langle e^{(n)}|e_{(\ell)}\rangle \xi^{\ell'} \eta^{(i)}{}_n$$
$$= |\eta_{(i)}\rangle \delta^n{}_\ell \xi^{\ell'} \eta^{(i)}{}_n$$
$$= |\eta_{(i)}\rangle \eta^{(i)}{}_\ell \xi^{\ell'}$$

$$|e_{(k)}\rangle \xi^{k'} = |e_{(n)}\rangle \eta^n_{(i)} \eta^{(i)}{}_\ell \xi^{\ell'} \qquad ③$$

$\Rightarrow \quad \eta^n_{(i)} \eta^{(i)}{}_\ell = \delta^n{}_\ell$

put in Σ – normalize:

$$\sum_i \eta^{n}_{(i)} \eta^{(i)}{}_{\ell} = \frac{\delta^n{}_\ell}{\langle \eta^{(i)}|\eta_{(i)}\rangle}$$

"completeness"/"closure" of basis set η.

vector, $i j k ..$ $\qquad$ components

Suppose we have 2 basis sets $|\eta_i\rangle$ and $|\gamma_j\rangle$

Expand ~~one~~ each in terms of the other

$$|\eta_{(i)}\rangle = |\gamma_{(j)}\rangle\, \eta^{j}{}_{(i)} \quad \text{and} \quad |\gamma_{(j)}\rangle = |\eta_{(k)}\rangle\, \gamma^{k}{}_{(j)} \qquad (1)$$

$$\downarrow$$

$$|\eta_{(i)}\rangle = |\gamma_{(j)}\rangle\, \mathbb{N}^{j}{}_{i} \qquad\qquad |\gamma_{(j)}\rangle = |\eta_{(k)}\rangle\, \mathbb{G}^{k}{}_{j}$$

$$= |\eta_{(k)}\rangle\, \mathbb{G}^{k}{}_{j}\, \mathbb{N}^{j}{}_{i}$$

$$= \sum_k \sum_j |\eta_{(k)}\rangle\, \mathbb{G}^{k}{}_{j}\, \mathbb{N}^{j}{}_{i}$$

$$= \sum_k |\eta_{(k)}\rangle \underbrace{\sum_j \mathbb{G}^{k}{}_{j}\, \mathbb{N}^{j}{}_{i}}_{\text{must be } \delta^{k}{}_{i}}$$

so, $$\sum_j \mathbb{G}^{k}{}_{j}\, \mathbb{N}^{j}{}_{i} = \delta^{k}{}_{i} \longrightarrow \text{matrix multiplication}$$

$$\mathbb{B}\mathbb{A} = \mathbb{I} \quad \text{or} \quad \mathbb{B} = \mathbb{A}^{-1} \qquad \mathbb{A} = \mathbb{B}^{-1}$$

$$\mathbb{G}\mathbb{N} = \mathbb{I} \qquad\qquad \mathbb{G} = \mathbb{N}^{-1} \qquad \mathbb{N} = \mathbb{G}^{-1}$$

notation is consistent $M^{i}{}_{j}$

ROW

COLUMN

note then, $M^{T}{}_{i}{}^{j} = M^{j}{}_{i}$

For duals — $\quad \langle \eta^{(i)}| = \sum_j \underbrace{(G^j{}_i)^*}_{\text{indices not right ... fix}} \langle \gamma^j|$

DEFINE $\quad (B^j{}_i)^* \equiv B_j^{*\,i} \equiv B^{\dagger i}{}_j$

so, $\quad \langle \eta^{(i)}| = \sum_j G_j^{*\,i} \langle \gamma^{(j)}| = G^{\dagger i}{}_j \langle \gamma^{(j)}|$

Take inner product $\quad \langle \gamma^{(k)}|\eta_{(i)}\rangle = \sum_j \underbrace{\langle \gamma^{(k)}|\gamma_{(j)}\rangle}_{\text{O.N.}} N^j{}_i$

$$= \sum_j \delta^k{}_j N^j{}_i = N^k{}_i$$

component of $\vec{\gamma}$ along $\vec{\eta}$

note $\quad \langle \eta^{(k)}|\eta_{(i)}\rangle = \delta^k{}_i = \sum_j \langle \eta^{(k)}|\gamma_{(j)}\rangle N^j{}_i$

$$= \sum_j \langle \gamma^{(j)}|\eta_{(k)}\rangle^* N^j{}_i$$

$$= (N^j{}_k)^* N^j{}_i$$

$$= N^{\dagger k}{}_j N^j{}_i = \delta^k{}_i$$

$\Rightarrow N^\dagger$ is defined to be right for matrix multiplication

so, $\quad N^\dagger N = I$ $\qquad N \quad G$

Further, when the bases are O.N. ~~A~~ and ~~B~~ are unitary:

$$N^\dagger = N^{-1} = G$$

$$|\eta_{(i)}\rangle = |\gamma_{(j)}\rangle \mathbf{N}^j{}_i$$
$$|\gamma_{(j)}\rangle = |\eta_{(i)}\rangle \mathbf{N}^{+i}{}_j$$

Now expand an arbitrary vector

$$|\xi\rangle = |\eta_{(i)}\rangle \xi^i$$

$$\underline{\underline{|\xi\rangle = |\gamma_{(j)}\rangle \xi^{j\prime}}}$$

substitute $\underline{\underline{\underline{|\xi\rangle = |\gamma_{(j)}\rangle \mathbf{N}^j{}_i \xi^i}}}$

and compare $=$ w/ $\equiv$

$$|\xi\rangle = |\gamma_{(j)}\rangle \xi^{j\prime} = |\gamma_{(j)}\rangle \mathbf{N}^j{}_i \xi^i \Rightarrow \quad \mathbf{N}^j{}_i \xi^i = \xi^{j\prime}$$

and $\quad \mathbf{N}^{+i}{}_j \xi^{j\prime} = \xi^i$.

the same operation that connects the bases, also ~~changes~~ transforms the components of a general vector.

A familiar example, ~~E_3~~ E_3

conventional way: $\quad \vec{x} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

now: $\quad |x\rangle = |\hat{e}_{(i)}\rangle x^i$.

Being careful w/ indices:

(A) $\quad |e_{(i)}\rangle = |e_{(j)}\rangle e^j{}_{(i)} \quad : \quad e^1_{(1)} = 1;\ e^2_{(1)} = 0;\ e^3_{(1)} = 0 \Rightarrow \hat{\imath}$

$$e^{(1)}_2 = 0;\ e^2_{(2)} = 1;\ e^3_{(2)} = 0 \Rightarrow \hat{\jmath}$$

etc.

which can be represented as matrices:

$$e_{(1)} = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad e_{(2)} = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad e_{(3)} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$e^{(1)} = (1,0,0) \qquad \text{etc.}$$

we can separate out the components of a vector, within a particular basis, by projecting.

$$\langle e^{(j)}|x\rangle = e^{(j)}_{\ \ell} \langle e^{(\ell)}|x\rangle \qquad \text{Ⓐ}$$

$$= e^{(j)}_{\ \ell} \langle e^{(\ell)}| e_{(i)}\rangle x^i$$

$$= \delta^{\ell}_{\ i}\, e^{(j)}_{\ \ell}\, x^i = e^{(j)}_{\ i}\, x^i = \sum_i e^{(j)}_{\ i} x^i$$

using familiar notation: $\langle e^{(1)}|x\rangle = \hat{i}\cdot\vec{x}$ LHS

$$= e^{(1)}_{\ 1} x^1 + e^{(1)}_{\ 2} x^2 + e^{(1)}_{\ 3} x^3 \quad \text{RHS}$$

$$= 1\cdot x^1 + 0.x^2 + 0.x^3$$

$$= x + 0.y + 0.z = x^1$$

In spirit of matrix notation

$$x^i = \begin{pmatrix}x^1\\x^2\\x^3\end{pmatrix}$$

$$\langle e^{(j)}|x\rangle = e^{(j)}_{\ i} x^i \Rightarrow \langle e^{(1)}|x\rangle = e^{(1)}_{\ i} x^i$$

$$= (1,0,0)\begin{pmatrix}x^1\\x^2\\x^3\end{pmatrix} = x^1$$

For 2 vectors $\vec{y}\cdot\vec{x}$

$$\langle y|x\rangle = y_j \langle e^{(j)}| e_{(i)}\rangle x^i$$

$$= \delta^j_{\ i}\, y_j x^i = y_i x^i \qquad (\Sigma)$$

$$= (y_1, y_2, y_3)\begin{pmatrix}x^1\\x^2\\x^3\end{pmatrix}$$

Consider an n-dimensional V_n. It is created by or consists of all linear combinations of n- linearly independent vectors → each set of n spans V_n.

Choose a subset $m<n$ w/ m all L.I. → spans another vector space $V_m \in V_n$.

The set of all vectors which are orthogonal to those spanning V_m creates an $V_m^{\perp}$ ⇒ (n-m) dimensions.

V_n is decomposed: $V_n = V_m \oplus V_m^{\perp}$

VIII. Similarly, 2 arbitrary spaces can be combined in a direct ~~product~~. sum:

$$V_{n+m} = V_n^a \oplus V_m^b$$

w/ properties:

A. $|\xi\rangle = |\eta(a)\rangle \oplus |\eta(b)\rangle$ w/ components dually noted

vectors combine

$(\eta(a);\eta(b))$

B. components combine

$$(\eta(a);\eta(b)) \oplus (\gamma(a);\gamma(b)) = (\eta(a)\oplus\gamma(a);\eta(b)\oplus\gamma(b))$$

IX. Direct ~~Sum~~ Product space: $V_{n \cdot m} = V_n^a \otimes V_m^b$

A. $|\eta(a)\rangle \otimes |\eta(b)\rangle$

B. $\left(c|\eta(a)\rangle \otimes |\eta(b)\rangle \right) = c\left(|\eta(a)\rangle \otimes |\eta(b)\rangle \right)$

C. $|\eta(a)\rangle \otimes \left(|\xi(b)\rangle \oplus |\gamma(b)\rangle \right) = |\eta(a)\rangle \otimes |\xi(b)\rangle \oplus$
$|\eta(a)\rangle \otimes |\gamma(b)\rangle$

D. suppose $|\alpha_{(i)}(a)\rangle$ is a basis set in V^a
$|\beta_{(j)}(b)\rangle$ " V^b

then $|\alpha_{(i)}(a)\rangle \otimes |\beta_{(j)}(b)\rangle$ forms an $n \times m$ dimensional basis set in $V_{n \cdot m}$.

expansion: $|\xi(a)\rangle = |\alpha_{(i)}(a)\rangle a^i = |\alpha_{(i)}(a)\rangle \xi^i_{[\alpha];a}$

$|f(b)\rangle = |\beta_{(j)}(b)\rangle b^j = |\beta_{(j)}(b)\rangle f^j_{[\beta];b}$

Product vector:

$$|\xi(a)\rangle \otimes |f(b)\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} |\alpha_{(i)}(a)\rangle \otimes |\beta_{(j)}(b)\rangle \xi^i_{[\alpha];a} f^j_{[\beta];b}$$

use notation $|\xi(a)\rangle \otimes |f(b)\rangle = |\xi(a) f(b)\rangle$

Inner product: $\langle \delta(a)\, \varepsilon(b) | \xi(a) f(b)\rangle =$
$\langle \delta(a)|\xi(a)\rangle \langle \varepsilon(b)|f(b)\rangle$

spaces stay separate

Lecture

The power of this for QM. comes from the ability to have ∞ sized vector spaces. $\rightarrow$ countably or continuously ∞.

To the other axioms, add:

X. A <u>compact space</u> is defined by:
Suppose for any vector $\xi \in V$ there is a series $|\eta_{(i)}\rangle$ with the property that there is at least one $\eta_{(i)}$ with

$$|\xi - \eta_{(i)}| < \varepsilon \quad \text{for any arbitrarily small } \varepsilon.$$

$\rightarrow$ physicists familiar w/ such spaces, most prominent:

set of square-integrable functions of a real variable, $\mathbb{L}^2(a,b)$

Suppose we have 2 functions $f(x)$ and $g(x)$ defined on $a \leq x \leq b$ for the continuous variable, x.

- combine them $\quad h(x) = f(x) + g(x) \rightarrow$ another function
- form scalar product $\quad (f,g) \equiv \int_a^b f^*(x)\, g(x)\, dx$

norm $\quad (f,f) = N_f = \int_a^b f^*(x) f(x)\, dx$

$$= \int_a^b |f(x)|^2 dx$$

if <u>finite</u> $\Rightarrow$ "square-integrable"

if $(f,g) = 0$; f & g are orthogonal functions