We have a scalar field now... generalize to an abstract quantity – but start to work in covariant formalism

$\varphi(\vec{x}, t)$: which could have more than one component (distinct from spacetime coordinates) → still transforms as a Lorentz scalar.

context implies spacetime

$$\varphi_i(\vec{x}, t) = \varphi_i(x^\mu) = \varphi_i(x)$$

Action:
$$S = \int_{t_1}^{t_2} dt \int_{\text{volume}} d^3x \, \mathcal{L}[\varphi_i, \dot{\varphi}_i, \partial\varphi_i/\partial x]$$
$$= \int_{t_1}^{t_2} dt \int_{\text{vol}} d^3x \, \mathcal{L}[\varphi_i, \partial_\mu \varphi_i]$$

The variation, appropriate for Hamilton's Principle

$$\delta\varphi_i = \varphi_i(x) - \varphi_i'(x)$$ vanishing at the spacetime boundary.

$$\delta S = \int dt \int d^3x \, \delta\mathcal{L} = 0$$ for physical system.

In covariant form

$$\delta\mathcal{L} = \sum_i \left\{ \frac{\delta\mathcal{L}}{\delta\varphi_i} \delta\varphi_i + \sum_\mu \frac{\delta\mathcal{L}}{\delta(\partial\varphi_i/\partial x^\mu)} \delta\left(\frac{\partial\varphi_i}{\partial x^\mu}\right) \right\}$$

Interchange $\frac{\partial}{\partial x^\mu}$ and δ

$$\frac{\partial}{\partial x^\mu}(\delta\varphi_i) = \frac{\partial}{\partial x^\mu}(\varphi_i(x) - \varphi_i'(x)) = \delta\left(\frac{\partial \varphi_i}{\partial x^\mu}\right)$$

~~Integrate the 2nd term in $\delta\mathcal{L}$ by parts~~

$$\int dt \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial\varphi_i/\partial x^\mu)} \frac{\partial}{\partial x^\mu}(\delta\varphi_i) = \int dt \int d^3x \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi^i/\partial x^\mu)}\delta\varphi_i\right]$$

$$- \int dt \int d^3x \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi^i/\partial x^\mu)}\right]\delta\varphi_i$$

Now integrate 1st term.

$$1^{st} = \int dt \int d^3x \frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi_i/\partial t)}\delta\varphi_i\right] - \int dt \int d^3x \frac{\partial}{\partial x^j}\left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi_i/\partial x^j)}\delta\varphi_i\right]$$

$$\int d^3x \left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi_i/\partial t)}\delta\varphi_i\right]_{t_1}^{t_2} = 0$$

like $\int d^3x\, \vec{\nabla}\cdot\vec{F} = \int \vec{F}\cdot d\vec{A} = 0$

since $\delta\varphi_i = 0$ @ surface

So,

$$\delta S = \int d^3x\, dt \sum_i \left\{\frac{\partial \mathcal{L}}{\partial \varphi_i} - \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi_i/\partial x^\mu)}\right]\right\}\delta\varphi_i = 0$$

and we get

$$\frac{\partial \mathcal{L}}{\partial \varphi_i} - \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial\varphi_i/\partial x^\mu)}\right] = 0 \qquad \forall i$$

which is the covariant form of E-L equations for scalar field

The "conjugate momentum field": $\pi_i(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_i}$

Can then construct the Hamiltonian density as before.

Remember, historically the first effort at a relativistic quantum wave equation was (by Schroedinger, Klein, Gordon -- different Gordon)

$$\cancel{H^2 \psi(x)} \quad H^2 \psi(x) = E^2 \psi = (p^2c^2 + m^2c^4)\psi(x)$$

using the (now called First) Quantization prescription:

$$H^2\psi(x) \xrightarrow{Q1} \left[c^2(-i\hbar\vec{\nabla})^2 + m^2c^4\right]\psi$$
$$= (-c^2\hbar^2\vec{\nabla}^2 + m^2c^4)\psi$$

rearranging

$$(-c^2\hbar^2\vec{\nabla}^2 - E^2 + m^2c^4)\psi = 0$$

further, $E \xrightarrow{Q1} i\hbar \frac{\partial}{\partial t}$

$$E^2 \xrightarrow{Q1} -\hbar^2 \frac{\partial^2}{\partial t^2}$$

$$(c^2\hbar^2\vec{\nabla}^2 - \hbar^2\frac{\partial^2}{\partial t^2} - m^2c^4)\psi(x) = 0$$

$$(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial}{\partial t^2} - \frac{m^2c^2}{\hbar^2})\psi = 0$$

a 2nd order wave equation

remember: $\partial^\mu \equiv [\partial^0, -\vec{\nabla}]$ $\qquad \partial_\mu \equiv [\partial/\partial t, +\vec{\nabla}]$

$$= [\tfrac{1}{c}\tfrac{\partial}{\partial t}, -\vec{\nabla}]$$

so $\qquad i\hbar\partial^\mu = [\frac{i\hbar}{c}\partial^0, -i\hbar\vec{\nabla}]$

and

$$\partial^\mu\partial_\mu \equiv \Box = \cancel{\ } = \partial^\mu\partial^\nu g_{\mu\nu}$$

$$\begin{aligned}\partial^\mu\partial_\mu &= \partial^0\partial^0 g_{00} + \partial^i\partial^j g_{ij} \\ &= \partial^0\partial^0(1) + \partial^i\partial^j(-1) \\ &= \partial^0\partial^0 - (-\vec{\nabla})\cdot(-\vec{\nabla}) = \partial^{0\,2} - \vec{\nabla}\cdot\vec{\nabla} \\ &= \frac{\partial}{\partial t^2} - \vec{\nabla}\cdot\vec{\nabla} = \Box\end{aligned}$$

and we can write

$$\left(\frac{1}{c^2}\frac{\partial}{\partial t^2} - \vec{\nabla}^2 + \frac{m^2c^2}{\hbar^2}\right)\psi(x) = 0$$

$$(\Box + m^2)\psi(x) = 0 \qquad \text{w/ } \hbar = c = 1$$

$$(\partial^\mu\partial_\mu + m^2)\psi(x) = 0$$

The covariant K-G equation.

This is what E-L equations must give

So, guided by our last example, we could propose

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2 \right)$$

$$= \frac{1}{2}\left(g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - m^2 \varphi^2 \right)$$

$$= \frac{1}{2}\left(\dot{\varphi}^2 - \vec{\nabla}\cdot\varphi \, \vec{\nabla}\cdot\varphi - m^2 \varphi^2 \right)$$

check:

$$\frac{\partial \mathcal{L}}{\partial \varphi} = -m^2 \varphi \qquad \frac{\partial \mathcal{L}}{\partial \varphi_i} - \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial \varphi_i / \partial x^\mu)} \right] = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial \varphi / \partial x^\eta)} = \frac{1}{2}\left\{ g^{\mu\eta} \frac{\partial \varphi}{\partial x^\mu} + g^{\eta\nu} \frac{\partial \varphi}{\partial x^\nu} \right\} \text{ general.}$$

$$= g^{\eta\nu} \frac{\partial \varphi}{\partial x^\nu}$$

so,

$$\frac{\partial}{\partial x^\eta}\left(\frac{\partial \mathcal{L}}{\partial(\partial \varphi / \partial x^\eta)} \right) = g^{\eta\nu} \frac{\partial^2 \varphi}{\partial x^\eta \partial x^\nu}$$

$$= g^{00} \frac{\partial^2 \varphi}{\partial x^0 \partial x^0} + g^{11} \frac{\partial^2 \varphi}{\partial x^1 \partial x^1} + \cdots$$

$$= \frac{\partial^2 \varphi}{\partial t^2} - \vec{\nabla}^2 \varphi$$

and from E-L:

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial \varphi / \partial x^\mu)} \right) = 0$$

$$-m^2 \varphi - \frac{\partial^2 \varphi}{\partial t^2} + \vec{\nabla}^2 \varphi = 0 \qquad \checkmark$$

The Hamiltonian density

$$\mathcal{H} = \pi\dot{\varphi} - \mathcal{L} \qquad \pi = \dot{\varphi} \quad \text{from } \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$
$$= \dot{\varphi}^2 - \mathcal{L}$$
$$= \dot{\varphi}^2 - \frac{1}{2}\left[\dot{\varphi}^2 - \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi - m^2\varphi^2\right]$$

$$\mathcal{H} = \frac{1}{2}\left[\dot{\varphi}^2 + \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi + m^2\varphi^2\right] \quad = \text{energy density.}$$

... a positive quantity. $\rightarrow$ distinct from Schroedinger 1st look.

The Hamiltonian : $\quad H = \int d^3x\, \mathcal{H}$

Investigation of symmetries is a particularly important task and is very convenient in the Lagrangian formalism

Consider first a change in coordinates which is infinitesimal:

$$x'_\mu = x_\mu + \delta x_\mu$$

This can induce changes in a function of x of 2 sorts. First, it can change the shape of the function

$$\phi'(x) = \phi(x) + \bar{\delta}(x)$$

$$\bar{\delta}\phi(x) = \phi'(x) - \phi(x) \qquad \text{("local variation")}$$

and the change of the function due to evaluation at the different x

$$\phi'(x') = \phi(x) + \delta\phi(x) \rightarrow \delta f(x) \Rightarrow$$ change of $f(x)$ for both shape and x

The two variations are related:

$$\bar{\delta}\phi(x) = \phi'(x) - \varphi(x)$$

$$= \varphi'(x) + \underline{\varphi'(x')} - \varphi'(x') - \underline{\varphi(x)}$$

$$= \underbrace{\varphi'(x) - \varphi'(x')}_{\text{same function @ } x \text{ and } x' \text{ which are close}} + \delta\varphi(x)$$

$$= \delta\varphi(x) - \frac{\partial\varphi'(x)}{\partial x_\mu}\delta x_\mu + \cdots$$ in Taylor expansions

$$\bar{\delta}\varphi(x) = \delta\varphi(x) - \frac{\partial\varphi'(x)}{\partial x_\mu}\delta x_\mu \xrightarrow{\text{lowest order}} \delta\varphi(x) - \frac{\partial\varphi(x)}{\partial x_\mu}\delta x_\mu$$

Now consider the results of demanding that the action be invariant wrt transformations above

$$\delta S \equiv \int_{\Omega'} d^4x' \mathcal{L}'(x') - \int_{\Omega} d^4x\, \mathcal{L}(x) \stackrel{!}{=} 0$$

notation: $\mathcal{L}(x) \equiv \mathcal{L}[\varphi(x), \partial\varphi(x)/\partial x_\mu]$
simplified to emphasize the $x \Rightarrow x'$ transformation

From above: $\delta \mathcal{L}(x) = \mathcal{L}'(x') - \mathcal{L}(x)$

$$\delta S = \int_{\Omega'} d^4x'\, \delta\mathcal{L}(x) + \int_{\Omega'} d^4x'\, \mathcal{L}(x) - \int_{\Omega} d^4x\, \mathcal{L}(x)$$

The Jacobian: $d^4x' = \dfrac{\partial(x'^\mu)}{\partial(x^\nu)} d^4x$

$$= \left(1 + \frac{\partial \delta x^\mu}{\partial x^\mu}\right) d^4x$$

again, to 1st order in expansion

$$\delta S = \int_{\Omega} d^4x\, \delta\mathcal{L}(x) + \int_{\Omega} d^4x\, \underbrace{\frac{\partial \delta x^\mu}{\partial x^\mu}\delta\mathcal{L}}_{\text{2nd order}} + \int_{\Omega} d^4x\, \frac{\partial \delta x^\mu}{\partial x^\mu}\mathcal{L}(x)$$

$$= \int_{\Omega} d^4x\, \delta\mathcal{L}(x) + \int_{\Omega} d^4x\, \frac{\partial \delta x^\mu}{\partial x^\mu}\mathcal{L}(x)$$

from $\bar{\delta}\mathcal{L}(x) = \delta\mathcal{L}(x) - \dfrac{\partial\mathcal{L}(x)}{\partial x^\mu}\delta x^\mu$

$$\delta S = \int_{\Omega} d^4x \left[\bar{\delta}\mathcal{L}(x) + \frac{\partial\mathcal{L}(x)}{\partial x^\mu}\delta x^\mu\right] + \int_{\Omega} d^4x\, \frac{\partial \delta x^\mu}{\partial x^\mu}\mathcal{L}(x)$$

$$\delta S = \int_\Omega d^4x \left[\bar{\delta} \mathcal{L}(x) + \frac{\partial}{\partial x^\mu} \left(\mathcal{L}(x) \delta x^\mu \right) \right]$$

Now, remembering that $\mathcal{L}(x) = \mathcal{L}\left[\varphi(x), \frac{\partial \varphi(x)}{\partial x_\mu} \right]$

So,

$$\bar{\delta} \mathcal{L}(x) = \frac{\partial \mathcal{L}(x)}{\partial \varphi} \bar{\delta} \varphi(x) + \frac{\partial \mathcal{L}(x)}{\partial(\partial^\mu \varphi)} \bar{\delta}(\partial^\mu \varphi)$$

$\bar{\delta}(\partial^\mu \varphi) = \partial^\mu \delta \varphi(x)$ can be shown!

$$\begin{aligned}
\bar{\delta} \mathcal{L}(x) &= \frac{\partial \mathcal{L}(x)}{\partial \varphi} \bar{\delta} \varphi(x) + \frac{\partial \mathcal{L}(x)}{\partial(\partial^\mu \varphi)} \partial^\mu (\bar{\delta} \varphi) \\
&= \frac{\partial \mathcal{L}}{\partial \varphi} \underline{\bar{\delta} \varphi(x)} + \partial^\mu \left(\frac{\partial \mathcal{L}(x)}{\partial(\partial^\mu \varphi)} \right) \bar{\delta} \varphi(x) \\
&\qquad - \partial^\mu \left(\frac{\partial \mathcal{L}(x)}{\partial(\partial^\mu \varphi)} \right) \underline{\bar{\delta} \varphi(x)} + \frac{\partial \mathcal{L}(x)}{\partial(\partial^\mu \varphi)} \partial^\mu (\bar{\delta} \varphi) \\
&= \left[\frac{\partial \mathcal{L}(x)}{\partial \varphi} - \partial^\mu \left(\frac{\partial \mathcal{L}(x)}{\partial(\partial^\mu \varphi)} \right) \right] \bar{\delta} \varphi(x) + \underline{\partial^\mu} \left[\frac{\partial \mathcal{L}(x)}{\partial(\partial^\mu \varphi)} \bar{\delta} \varphi \right] \Big\}
\end{aligned}$$

$$\delta S = \int_\Omega d^4x \left\{ \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)} \right) \right] \bar{\delta} \varphi + \frac{\partial}{\partial x_\mu} \left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)} \bar{\delta} \varphi + \mathcal{L}(x) \delta x^\mu \right] \right\}$$

If $\delta S = 0$ for an invariant action, then the $\{ \ \} = 0$.

If $\varphi(x)$ satisfies an equation of motion, then

$$\underbrace{[\quad]}_{\substack{\text{Euler-Lagrange} \\ \text{Equation}}} \bar{\delta} \varphi = 0$$

So, the condition for an invariant action becomes.

$$\frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)}\,\bar{\delta}\varphi + \mathcal{L}(x)\,\delta x^\mu\right] = 0$$

$$\Big\downarrow$$

$$\delta\varphi(x) - \frac{\partial \varphi(x)}{\partial x_\mu}\,\delta x_\mu$$

$$\frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)}\,\delta\varphi(x) - \frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)}\,\frac{\partial \varphi}{\partial x_\mu}\,\delta x_\mu + \mathcal{L}(x)\,\delta x^\mu\right] = 0$$

$$\frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)}\,\delta\varphi(x) - \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)}\,\frac{\partial \varphi}{\partial x^\nu} - g_{\mu\nu}\mathcal{L}\right)\delta x^\nu\right] = 0$$

This looks like an equation of continuity – 4-divergence of something $= 0$

Define $\quad J_\mu(x) = [\qquad\qquad]$ so,

we have $\quad \partial^\mu J_\mu(x) = 0 \quad$ as the condition for an ~~const~~ invariant action.

$$\delta S = 0 = \int d^4x\,\frac{\partial}{\partial x_\mu}\,J_\mu(x)$$

look at 3-space integration

$$= \int d^3x\,\frac{\partial}{\partial x_\mu}\,J_\mu(x)$$

$$= \int_V d^3x\,\frac{\partial}{\partial x_0}\,J_0(x) + \int_V d^3x\,\vec{\nabla}\cdot\vec{J}(x)$$

$\parallel$ Gauss' Theorem

$$\oint_V d\vec{A}\cdot\vec{J}(x) \to 0 \text{ at } \infty$$

So, then we have

$$\delta S = 0 = \int dt \int d^3x \, \frac{\partial}{\partial t} J_0(x)$$

$$= \int dt \, \frac{\partial}{\partial t} \underbrace{\int d^3x \, J_0(x)}_{\text{"}Q\text{"}}$$

$$\delta S = 0 = \int dt \, \frac{\partial Q}{\partial t} \qquad Q \text{ is \underline{constant}}$$

These notions are suggestive enough of electromagnetism, that the following vocabulary is used:

$J_\mu(x)$ is a <u>current</u> -- abstract quantity at this time, not necessarily related to electric ~~charge~~ current.

Q is a charge.

The divergencelessness (whew!) of the 4-current, J_μ leads to a constant charge.

constant Q

→ a conservation law ^ related to the invariance of the action under an infinitesimal transformation

invariance of $\mathcal{L}$

think Lie Group

Noether's Theorem: Every continuous transformation leaving the Lagrangian invariant, has associated with it a conserved current and a conservation law.

Remember, a conservation law in quantum mechanics implies an operator which is a constant of the motion.

Emmy Noether showed an amazing and absolutely fundamental thing:

the "Noether charge" is BOTH → constant of the motion.
→ generator of the Lie group which covers the infinitesimal transformation