We have found it necessary to introduce 2 ordering operations:

1) Normal ordering – to make the vacuum have zero energy
2) Chronological product. – to be able to make the S-matrix expansion usable.

I introduced "P" for the Hamiltonian in the interaction representation, but I need to modify it slightly for individual field operators.

Write an arbitrary field operator

including annihilation

$$\alpha(x) \equiv A(x) + C(x) \qquad \beta(x) = A'(x) + C'(x)$$

including creation.

$$\bar{\alpha}(x) \equiv \bar{A}(x) + \bar{C}(x)$$

The T operation, or Wick Time Ordering Operator is defined

$$T[\alpha(x)\beta(y)] \equiv \alpha(x)\beta(y) \qquad x_0 > y_0$$
$$= P_{\alpha\beta}\,\beta(y)\alpha(x) \qquad x_0 < y_0$$

where $P_{\alpha\beta} = -1$ if fermions
$= +1$ if bosons.

19 215 This is for pairs. In general, get a factor $(-1)^P$ where $P = \#$ permutations of fermion operators.

This has no real practical effect since fermion operators always appear in pairs!

Normal ordered: ($C \to L$, $A \to R$)

① $\alpha(x)\beta(y) = A(x)A'(y) + \underline{A(x)C'(y)} + C(x)A'(y) + C(x)C'(y)$

$:\alpha(x)\beta(y): \; = A(x)A'(y) + P_{\alpha\beta}\, C'(y)A(x) + C(x)A'(y) + C(x)C'(y)$

(use prime to indicate "from β")

Add and subtract $C'(y)A(x)$

$$\alpha(x)\beta(y) = A(x)A'(y) + A(x)C'(y) + C(x)A'(y) + C(x)C'(y) + C'(y)A(x) - C'(y)A(x)$$

* if fermions:

$:\alpha(x)\beta(y): \; = A(x)A'(y) - C'(y)A(x) + C(x)A'(y) + C(x)C'(y)$

$= AA' - C'A + CA' + CC'$

so ① $\alpha(x)\beta(y) = \; :\alpha(x)\beta(y): + C'(y)A(x) + A(x)C'(y)$

$\underline{AA'} + AC' + \underline{CA'} + \underline{CC'} = \underline{AA'} - C'A + \underline{CA'} + \underline{CC'} \;\; (+ C'A + AC')$

$= \; :\alpha(x)\beta(y): + \{A(x), C'(y)\}$ Ⓕ

* if bosons:

#'s

$\alpha(x)\beta(y) = \; :\alpha(x)\beta(y): + [A(x), C'(y)]$ Ⓑ

$= AA' + C'A + CA' + CC'$

take Vacuum Expectation Value, VEV. –

$\langle 0|(1)|0\rangle$ $\langle 0|\alpha(x)\beta(y)|0\rangle = \cancel{\langle 0|A(x)A'(y)|0\rangle}^{\,0} + \langle 0|A(x)\underline{C'(y)}|0\rangle$
$+ \cancel{\langle 0|C(x)A'(y)|0\rangle}_{\,0} + \cancel{\langle 0|C(x)C'(y)|0\rangle}_{\,0}$

$= \langle 0|A(x)C'(y)|0\rangle$

add zero $= -\langle 0|C'A|0\rangle + \langle 0|C'A|0\rangle$

rewrite for fermions

$$\langle 0|A(x)C'(y)|0\rangle = -\underbrace{\langle 0|C'(y)A(x)|0\rangle}_{=0 \text{ explicitly}} + \langle 0|C'(y)A(x)|0\rangle \;(=0)$$
$$+ \langle 0|\underline{A(x)C'(y)}|0\rangle$$
$$= 0 + \langle 0|\{A(x), C'(y)\}|0\rangle$$
$$= \{A(x), C'(y)\}$$

or for bosons

$$= 0 + \langle 0|[A(x), C'(y)]|0\rangle$$
$$= [A(x), C'(y)]$$

Then our fermion statement (F) can be written

$$= :\alpha\beta: + \langle 0|\{A, C'\}|0\rangle = :\alpha\beta: + \{A, C'\}$$

$$\alpha(x)\beta(y) = :\alpha(x)\beta(y): + \langle 0|A(x)C'(y)|0\rangle$$

But $\langle 0|:\text{anything}:|0\rangle = 0$, so

so $\langle 0|\alpha(x)\beta(y)|0\rangle = \{A(x), C'(y)\}$

and

(B) $\alpha(x)\beta(y) = :\alpha(x)\beta(y): + \langle 0|A(x)C'(y)|0\rangle$ same

So $\langle 0|\alpha(x)\beta(y)|0\rangle = [A(x), C'(y)]$

Suppose we changed the normal product ordering — the other time relation:

$$:\beta(y)\alpha(x): \;=\; A'(y)A(x) + C'(y)A(x) + :A'(y)C(x): + C'(y)C(x)$$
$$= A'(y)A(x) + \underline{C'(y)A(x)} + \underline{P_{\alpha\beta}C(x)A'(y)} + C'(y)C(x)$$

permute $\underline{\text{some}}$ back

$$= P_{\alpha\beta}A(x)A'(y) + \underline{C'(y)A(x)} + \underline{P_{\alpha\beta}C(x)A'(y)}$$
$$+ P_{\alpha\beta}C(x)C'(y)$$

(F) $\quad :\beta\alpha: \;=\; -AA' + C'A - CA' - CC'$
$\quad = -:\alpha\beta:$

(B) $\quad :\beta\alpha: \;=\; AA' + C'A + CA' + CC'$
$\quad = +:\alpha\beta:$

so

$$:\beta\alpha: = P_{\alpha\beta}:\alpha\beta:$$

Using the definition of the T product

BUT

either $\quad T(:\alpha\beta:) = :\alpha\beta: \quad$ OR $\quad = P_{\alpha\beta}:\beta\alpha: = P_{\alpha\beta}P_{\alpha\beta}:\alpha\beta:$

so as I said

$$T(:\alpha\beta:) = :\alpha\beta: \quad \text{-- statistics independent}$$

SO,

(F) or (B) $\quad T(\alpha\beta) = T(:\alpha\beta:) + T(\langle 0|\alpha\beta|0\rangle)$ a number

$\quad = :\alpha\beta: + T(\langle 0|\alpha\beta|0\rangle) \qquad \Delta$

already

Remember.. the whole point of $: :$ is that

$$\langle 0 | : \alpha\beta : | 0 \rangle = 0$$

Then

in Δ

$$\langle 0 | T(\alpha\beta) | 0 \rangle = T(\langle 0 | \alpha\beta | 0 \rangle)$$

$$\equiv \underbracket{\alpha\beta}$$ called a "contraction"

— not an operator.

so $T(\alpha\beta) = :\alpha\beta: + \underbracket{\alpha\beta}$ $\quad\Delta$

Products of 3 can be done .. $\alpha(x)\beta(y)\gamma(z)$

$T(\alpha\beta\gamma)$ specific case x_0 & $y_0 > z_0$

$$\begin{aligned} T(\alpha\beta\gamma) &= T(\alpha\beta)\gamma \\ &= :\alpha\beta:\gamma + T\langle 0|\alpha\beta|0\rangle\gamma \\ &= :\alpha\beta:\gamma + \underbracket{\alpha\beta}\,\gamma \end{aligned}$$

⋮ lots of algebra.

$$= :\alpha\beta\gamma: + :\underbracket{\alpha\beta}\gamma: + :\underbracket{\alpha\beta\gamma}: + :\alpha\underbracket{\beta\gamma}:$$

all symmetric permutations of the $\alpha\beta\gamma$ combination — independent of the specific x_0 & $y_0 > z_0$ conditions.

→ true for all time sequencing

using the definition of the T. product... for any pair of operators,

$$T(:\alpha\beta:) \nearrow :\alpha\beta: \qquad T(:\alpha\beta:) \searrow P_{\alpha\beta} :\beta\alpha: \; = P_{\alpha\beta} P_{\alpha\beta} :\alpha\beta:$$

statistics-independent

$$T(:\alpha\beta:) = :\alpha\beta:$$

so, T-ordering our statistics-independent statement... B_1 or B_2

$$T(\alpha\beta) = T(:\alpha\beta:) + T(\langle 0|\alpha\beta|0\rangle) \leftarrow \text{number}$$

$$T(\alpha\beta) = :\alpha\beta: + T(\langle 0|\alpha\beta|0\rangle)$$

Remember, the whole point of : : is that

$$\langle 0|:\alpha\beta:|0\rangle = 0 \quad \text{, no creation } \langle 0| \rightarrow \quad \leftarrow |0\rangle$$

taking VEV:

$$\langle 0|T(\alpha\beta)|0\rangle = T\langle 0|\alpha\beta|0\rangle$$

$$\equiv \underbracket{\alpha\beta} \qquad \text{"contraction"}$$

it's not an operator.

stop

3 operators can be done...

$T(\alpha\beta\gamma)$ -- consider special case $\alpha(x)\beta(y)\gamma(z)$

where x_0 and $y_0 > z_0$. ← specific case

$$T(\alpha\beta\gamma) = T(\alpha\beta)\gamma$$

$$= :\alpha\beta:\gamma + T\langle 0|\alpha\beta|0\rangle\gamma$$

$$= :\alpha\beta:\gamma + \underbracket{\alpha\beta}\gamma$$

$$T(\alpha\beta\gamma) = :\alpha\beta:\gamma + \underbracket{\alpha\beta}:\gamma: = \overset{①}{:\alpha\beta:\gamma} + \overset{②}{:\alpha\beta\gamma:}$$

look at this

85

① $\underline{:\alpha\beta:\gamma} = \overset{1.1}{\underline{:\alpha\beta: C''(z)}} + \overset{1.2}{\underline{\underbrace{:\alpha\beta: A''(z)}}}$

already normal ordered $\overset{so}{=}$ $:\alpha\beta\, A(z):$

~~look at this~~

(1.2) $C(x)C'(y)A''(z) + C(x)A'(y)A''(z)$

$+ P_{\alpha\beta}\, C'(y)A(x)A''(z) + A(x)A''(y)A''(z)$

look at this

(1.1) $\underline{:\alpha\beta: C''(z)} = \left[C(x)C'(y) + C(x)A'(y) + P_{\alpha\beta}\, C'(y)A(x) + A(x)A'(y)\right] C''(z)$

$$= \underset{1.1.1}{C(x)C'(y)C''(z)} + \underset{1.1.2}{\underline{C(x)A'(y)C''(z)}}$$

$$+ \underset{1.1.3}{\underline{P_{\alpha\beta}\, C'(y)A(x)C''(z)}} + \underset{1.1.4}{\underline{A(x)A'(y)C''(z)}}$$

look at this

later, look at this

1.1.2

$\underline{C(x)A'(y)C''(z)} = C(x)A'(y)C''(z) + \overbrace{C(x)C''(z)A'(y) - C(x)C''(z)A'(y)}^{0}$

$= C(x)\left(A'(y)C''(z) + C''(z)A'(y) - C''(z)A'(y)\right)$ 1.1.2*

$= C(x)\left(\{A'(y), C''(z)\} - C''(z)A'(y)\right)$

OR $= C(x)\left([A'(y), C''(y)] + C''(z)A'(y)\right)$

in any case, remember that

$\langle 0|\beta\gamma|0\rangle = [A'(y), C''(z)]$

OR $\langle 0|\beta\gamma|0\rangle = \{A'(y), C''(z)\}$

Remember we constructed a situation which ~~was~~ time-ordered in $\beta\gamma$... so,

$$\cancel{\langle 0|\beta\gamma|0\rangle = \langle 0|T(\beta\gamma)|0\rangle = \underset{\sqcup}{\beta\gamma}}$$

$$T(\beta\gamma) = :\beta\gamma: + T\langle 0|\beta\gamma|0\rangle$$

$$\beta\gamma = :\beta\gamma: + \underset{\sqcup}{\beta\gamma}$$

$$\cancel{\langle 0|\beta\gamma|0\rangle = 0 + \underset{\sqcup}{\beta\gamma}}$$

directly ~~and~~

$$\beta\gamma = \cancel{A'A''} + A'C'' + \cancel{C'A''} + \cancel{C'C''} = \cancel{A'A''} + P_{\beta\gamma}\, C''A' + \cancel{C'A''} + \cancel{C'C''} + \underset{\sqcup}{\beta\gamma}$$

$$A'(y)C''(z) = P_{\beta\gamma}\, C''(z)A'(y) + \underset{\sqcup}{\beta\gamma}$$

so,

1.1.2
$$C(x)A'(y)C''(z) = C(x)\left(P_{\beta\gamma}\, C''(z)A'(y) + \underset{\sqcup}{\beta\gamma}\right)$$

The "later" term:

same but x-z

1.1.3
$$P_{\alpha\beta}\, C'(y)A(x)C''(z) = P_{\alpha\beta}\, C'(y)\left(P_{\alpha\gamma}\, C''(z)A(x) + \underset{\sqcup}{\alpha\gamma}\right)$$

and last term:

1.1.4
$$A(x)A'(y)C''(z) = A(x)\left(P_{\beta\gamma}\, C''(z)A'(y) + \underset{\sqcup}{\beta\gamma}\right)$$

do it again $A(x) \rightarrow$

1.1.4
$$A(x)A'(y)C''(y) = P_{\beta\gamma}\left(P_{\alpha\gamma}\, C''(z)A(x)A'(y) + \underset{\sqcup}{\alpha\gamma}\, A'(y)\right) + A(x)\underset{\sqcup}{\beta\gamma}$$

So, we get:

$$T(\alpha\beta\gamma) = \overset{①}{:\alpha\beta:\gamma} + \overset{②}{\underbrace{\alpha\beta}\gamma}$$

$$= \underset{(1.1)}{:\alpha\beta: C''(z)} + C(x)C'(y)A''(z) + C(x)A'(y)A''(z)$$
$$+ \underset{(1.2)}{P_{\alpha\beta}\, C'(y)A(x)A''(z) + A(x)A'(y)A''(z)}$$

$$= C(x)C'(y)C''(z) \quad 1.1.1$$
$$+ P_{\beta\gamma} C(x) C''(z) A'(y) + C(x)\,\underbrace{\beta\gamma}: \qquad — \quad (1.1.2)$$
$$+ P_{\alpha\beta} P_{\alpha\gamma}\, C'(y) C''(z) A(x) + P_{\alpha\beta}\, C(y)\, \underbrace{\alpha\gamma} \qquad (1.1.3)$$
$$+ P_{\beta\gamma} P_{\alpha\gamma}\, C''(z) A(x) A'(y) + P_{\beta\gamma}\, \underbrace{\alpha\gamma}\, A'(y) + A(x)\,\underbrace{\beta\gamma} \quad (1.1.4)$$
$$+ C(x)C'(y)A''(z) + C(x)A(y)A(z) + P_{\alpha\beta}\, C(y) A(x) A''(z)$$
$$+ A(x)A'(y)A''(z) + \qquad (1.2)$$

$$+ \underbrace{\alpha\beta}\gamma \quad ②$$

— terms are $:\alpha\beta\gamma:$.

═ terms are $C(x)\,\underbrace{\beta\gamma} + A(x)\,\underbrace{\beta\gamma}$

≡ terms are $P_{\alpha\beta}\, C'(y)\, \underbrace{\alpha\gamma} + P_{\beta\gamma}\, \underbrace{\alpha\gamma}\, A'(y) = :\underbrace{\alpha\beta\gamma}:$

So,

$$T(\alpha\beta\gamma) = :\alpha\beta\gamma: + :\underbrace{\alpha\beta}\gamma: + :\underbrace{\alpha\beta\gamma}: + :\alpha\underbrace{\beta\gamma}:$$

all are symmetric permutations of the $\alpha\beta\gamma$ contractions

This is a general theorem, important for S matrix physics.

$\boxed{\text{Wick's Theorem}}$

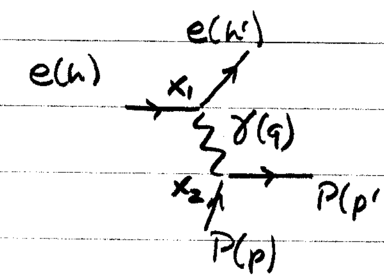
$$T(ABC \cdots XYZ) = \,:ABC\cdots XYZ:$$
$$+ \,:\underbrace{AB}C\cdots XYZ: + :AB\underbrace{C}\cdots XY\underbrace{Z}: \; + \text{ etc}$$

+ ... all possible pairs   + all double combinations   + all additional pair combinations.

until everything is contracted

WHY? YOU MIGHT ASK...

Imagine we're considering the process $e(k)P(p) \to e(k')P(p')$



$$|i\rangle = |e(k)\, P(p)\rangle$$
$$|f\rangle = |e(k')\, P(p')\rangle$$

$$\mathcal{L}_I(x) = -e\,\overline{\Psi}(x)\gamma^\mu \Psi(x) A_\mu(x) \qquad \text{and from the}$$

S matrix expansion we're obligated to evaluate

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d^4x_1 \int_{-\infty}^{\infty} d^4x_2 \cdots \int_{-\infty}^{\infty} d^4x_n \, P\left[\mathcal{L}_I(x_1) \mathcal{L}_I(x_2) \cdots \mathcal{L}_I(x_n) \right]$$

for our problem

$$P\left[\mathcal{L}_I(x_1) \mathcal{L}_I(x_2) \right] = \begin{cases} \mathcal{L}_I(x_1) \mathcal{L}_I(x_2) & x_{10} > x_{20} \\ \mathcal{L}_I(x_2) \mathcal{L}_I(x_1) & x_{20} > x_{10} \end{cases}$$

it's conditional.

One would just do it -- keep track of all time ordering and explicitly calculate (that's what Sakurai does).

However, Wick's Theorem greatly simplifies things as it can very quickly become unreasonably complicated otherwise.

It's a bit of magic -- the P product and the T product are the same for bosons and for <u>pairs</u> of fermion operators.

So, the conditional, complicated P expansion is simplified to a sum of terms.

Here's how it simplifies...

consider $\alpha(x)\beta(y) = \,:\alpha(x)\beta(y): + \langle 0|A(x)C'(y)|0\rangle$

$= \,:\alpha(x)\beta(y): + \{A(x), C'(y)\}$

a general fermion statement

- Suppose $\beta(y) \equiv \alpha(y)$ same field op as $\alpha(x)$, at y instead of x

$\alpha(x)\alpha(y) = \,:\alpha(x)\alpha(y): + \{A(x), C'(y)\}$

$\downarrow$ corresponds to

$\{a, b^\dagger\} = 0$ so,

$\alpha(x)\alpha(y) = \,:\alpha(x)\alpha(y):$

so $\langle 0|\alpha(x)\alpha(y)|0\rangle = \langle 0|:\alpha(x)\alpha(y):|0\rangle = 0$ by definition

then $T\langle 0|\alpha(x)\alpha(y)|0\rangle = \underbrace{\alpha\alpha} = 0$

- Suppose $\beta(y) \equiv \bar{\alpha}(y)$

$\alpha(x)\bar{\alpha}(y) = \,:\alpha(x)\bar{\alpha}(y): + \{A(x), \bar{C}'(y)\}$

$\downarrow$ corresponds to

$\{a, a^\dagger\} \neq 0$

so $\underbrace{\alpha\bar{\alpha}} \neq 0$

similarly $\underbrace{\bar{\alpha}\alpha} = 0$, $\underbrace{\alpha\phi} = 0$

$\longrightarrow$ this will reduce the number of terms by a considerable amount.

So, inside the Smatrix we had for $S^{(2)}$

$$P\left[\ \bar{\psi}(x_1)\gamma^\mu\psi(x_1)A_\mu(x_1)\ \bar{\psi}(x_2)\gamma^\nu\psi(x_2)A_\nu(x_2)\right]$$

which is $= \quad T\left[\qquad\qquad\qquad\right]$

which, from Wick's Theorem becomes,

$$= \gamma^\mu_{ij}\gamma^\nu_{lm}\, T\Big[\ \bar{\psi}_i(x_1)\underbrace{\psi_j(x_1)A_\mu(x_1)\,\bar{\psi}_l(x_2)\psi_m(x_2)A_\nu(x_2)}_{6\text{ operators}}\Big]$$

matrix indices, so can move them around.

$$= \gamma^\mu_{ij}\gamma^\nu_{lm}\sum_W \mathcal{O}^{(W)}_{\mu\nu}$$

the sum of operator contractions in the Wick expansion — $:\bar{\psi}\psi A\bar{\psi}\psi A:$ + all contractions. >50 terms.

BUT, from above since

$$\underbrace{\psi\psi} = \underbrace{\psi A} = \underbrace{\bar{\psi}A} = 0$$

∴ 8 terms are left which are non-zero.