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Here is a review of my conventions.

1 Metric, *et al.*

The Minkowski tensor is $g_{\mu\nu} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \text{-1} & 0 & 0 \\ 0 & 0 & \text{-1} & 0 \\ 0 & 0 & 0 & \text{-1} \end{pmatrix}$ having the "signature" $(1, -1, -1, -1)$. Conventionally, we refer to spacetime coordinates as **contravariant** 4-vectors, tensors of the 1^{st} rank, with the index *up*:

$$x^\mu = (x^0, x^1, x^2, x^3) \tag{1}$$
$$= (ct, x, y, z) \tag{2}$$
$$= [x^0, \vec{x}] \text{ or } (x^0, \mathbf{x}) \tag{3}$$

The context will delineate when the $(\ldots)$ is designating the contents of a fourvector or a real function. Of course, "blackboard units" will generally prevail in which $c = \hbar = 1$.

We **contract** with the metric tensor, $g_{\mu\nu}$ to change from contravariant to covariant tensors:

$$T_\mu = g_{\mu\nu} x^\nu = \sum_{\nu=0}^{3} g_{0\nu} x^\nu \tag{4}$$
$$T_0 = g_{00} x^0 = \sum_{\nu=0}^{3} g_{0\nu} x^\nu \tag{5}$$
$$T_1 = \sum_{\nu=0}^{3} g_{1\nu} x^\nu \tag{6}$$
$$= g_{11} x^1 = (-1) x^1 = -x \text{ etc} \tag{7}$$

So, this means that

$$x_\mu = (t, -x, -y, -z) \tag{8}$$
$$= (x_0, x_1, x_2, x_3) \tag{9}$$
$$= [x_0, -\vec{x}] \text{ or } (x_0, -\mathbf{x}) \tag{10}$$

Using the Einstein summation convention

$$g_{\mu\nu}x^{\nu} = x_{\mu} \tag{11}$$

The inverse of $g_{\mu\nu}$ is $g^{\mu\nu}$, so $g_{\mu\nu}g^{\mu\nu} = 1$ which implies the double summation $\sum_{\nu=0}^{3}\sum_{\mu=0}^{3}$. So also, then, $g^{\mu\nu} = g_{\mu\nu}$, as is obvious from the matrix representation. Conventionally, then

$$g^{\mu\nu}g_{\nu\alpha} = g_{\alpha}^{\mu} \equiv \delta_{\nu}^{\mu} \tag{12}$$

which is the 4-dimensional Kronecker delta "function" which has 1's on the diagonal. It's instructive to put this together element by element:

$$\begin{aligned} g^{\mu\nu}g_{\nu\alpha} &= \sum_{\nu=0}^{3} g^{\mu\nu}g_{\nu\alpha} \\ &= g^{00}g_{0\alpha} + g^{01}g_{1\alpha} + g^{02}g_{2\alpha} + g^{03}g_{3\alpha} \end{aligned} \tag{13}$$

In more detail:

$$\begin{aligned} g_{\alpha}^{0} &= g^{00}g_{0\alpha} + g^{01}g_{1\alpha} + \ldots \\ &= g^{00}g_{0\alpha} + 0 + 0 + 0 + \\ &= 1 \times g_{0\alpha} \quad \text{for } \alpha = 0 \text{ or } \quad = 0 \text{ for } \alpha \neq 0 \end{aligned} \tag{14}$$

$$\begin{aligned} g_{\alpha}^{1} &= 0 + g^{11}g_{1\alpha} + 0 + 0 \\ &= -1 \times g_{1\alpha} = (-1)(-1) = 1 \quad \text{for } \alpha = 1 \text{ or } \quad = 0 \text{ for } \alpha \neq 1 \end{aligned} \tag{15}$$

Generally, for any tensors of any rank,

$$A_{\mu\nu} = g_{\mu\alpha}g^{\nu\beta}A^{\alpha\beta} \text{ etc.} \tag{16}$$

The 4-dimensional dot product, or "inner product" is

$$\begin{aligned} A^{\mu}B_{\mu} &= A^{0}B_{0} + A^{1}B_{1} + A^{2}B_{2} + A^{3}B_{3} \\ &= A^{\mu}B^{\alpha}g_{\alpha\mu} \\ &= A^{0}B^{\alpha}g_{\alpha 0} + A^{1}B^{\alpha}g_{\alpha 1} + \ldots \\ &= A^{0}B^{0}g_{\alpha 0} + A^{1}B^{1}g_{11} + \ldots \\ &= A^{0}B^{0} - A^{1}B^{1} - A^{2}B^{2} - A^{3}B^{3} \\ &= A^{0}B^{0} - \vec{A}\cdot\vec{B} \\ &= A \cdot B \end{aligned} \tag{17}$$

...which is a scalar. The context for the use of the "dot" symbols will be obvious, whether 4-d or 3-d.

A "scalar" is an object which can be measured with a scale...doesn't depend on a coordinate system. Then, a "scalar field" is a function of coordinates which itself doesn't change under coordinate transformation.

2 Coordinate Transformations

Consider two neighboring points, one point, A, at spacetime coordinate x^{μ} and B at $x^{\mu} + dx^{\mu}$ and another coordinate system defined in terms of the first one, the "old" coordinates.

$$x'^{\mu} = f^{\mu}(x^{\mu}) \tag{18}$$

Then

$$
\begin{aligned}
dx'^{\mu} &= \frac{\partial f^{\mu}(x^{\mu})}{\partial x^{\nu}} dx^{\nu} \\
&= \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu} \\
dx'^{\mu} &\equiv \Lambda^{\mu}_{\nu} dx^{\nu}
\end{aligned}
\tag{19}
$$

The relation expressed as Eq. 19 is a defining relation for any coordinate transformation. Any quantity that transforms like a differential coordinate transformation, i.e., like 19 is a **4-vector**, a contravariant vector.

Now consider a scalar function of coordinates, $\phi(x^{\mu})$ and form the gradient of this function:

$$
A_{\nu} \equiv \frac{\partial \phi(x^{\mu})}{\partial x^{\nu}} \tag{20}
$$

Notice the position of the spacetime indices in the numerator and denominator of Eq. 20. By the rules of differentiation,

$$
\begin{aligned}
\frac{\partial \phi(x^{\mu})}{\partial x'^{\nu}} &= A'_{\nu} = \frac{\partial \phi(x^{\mu})}{\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \\
A'_{\nu} &= \frac{\partial x^{\sigma}}{\partial x'^{\nu}} A_{\sigma}
\end{aligned}
\tag{21}
$$

Look at the position of the primes in Eqs. 20 and **??**— the quantity in front of A_{σ} in Eq. **??** is *not* Λ^{σ}_{ν}. This is a different transformation equation, that of the gradient, which is a **contravariant** vector. If we contract contravariant and covariant vectors, and transform them:

$$
\begin{aligned}
A_{\mu} B^{\mu} \to A'_{\mu} B'^{\mu} &= A_{\nu} \frac{\partial x^{\nu}}{\partial x'^{\mu}} B^{\alpha} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \\
&= A_{\nu} B^{\alpha} \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \\
&= A_{\nu} B^{\alpha} \frac{\partial x^{\nu}}{\partial x^{\alpha}}
\end{aligned}
\tag{22}
$$

Now, $\frac{\partial x^{\nu}}{\partial x^{\alpha}} = \delta^{\nu}_{\alpha}$, so $A'_{\mu} B'^{\mu} = A_{\nu} B^{\alpha} \delta^{\nu}_{\alpha} = A_{\nu} B^{\nu}$. That the transformed product has the same symbolic form in either coordinate system, indicates that it is a scalar product.

2.1 The Lorentz Transformation

The transformation tensor, *Lambda*, has 16 components, but with the constraint that the interval must be invariant. This makes the transformation consistent with relativity. So, $ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$ is unchanged under the transformation:

$$
\begin{aligned}
ds'^2 &= ds^2 \\
g_{\mu\nu} dx'^{\mu} dx'^{\nu} &= g_{\mu\nu} dx^{\alpha} dx^{\beta} \\
g_{\mu\nu} \Lambda^{\mu}_{\rho} dx^{\rho} \Lambda^{\nu}_{\xi} dx^{\xi} &= g_{\mu\nu} \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\xi} dx^{\rho} dx^{\xi} \\
&= g_{\alpha\beta} dx^{\alpha} dx^{\beta} \\
&= g_{\rho\xi} dx^{\rho} dx^{\xi}
\end{aligned}
\tag{23}
$$

The trick in Eq. 23 reflects the fact that the indices in the previous equation are just dummies, since they are summed over, and can be labeled anything. Keeping in mind that since the indices are explicitly noted, these terms are all just numbers and can be moved around at will. So,

$$
\Lambda^{\mu}_{\rho} g_{\mu\nu} \Lambda^{\nu}_{\xi} = g_{\rho\xi} = \Lambda^{T\mu}{}_{\rho} g_{\mu\nu} \Lambda^{\nu}_{\xi} \tag{24}
$$

Doing the transpose flips the positions of the indices. Further, contracting gives $\Lambda_{\nu\rho}\Lambda^{\nu}_{\xi} = g_{\rho\xi}$ which is called "pseudo orthogonality". This is the equivalent of 10 conditions on the 16 components of the Lorentz Transformation, leaving 6 independent parameters: 3 components of relative velocity and 3 components of angle relating the $\vec{x}$ and the $\vec{x'}$ space axes. For zero relative angles, then the transformations are the familiar Pure Lorentz Transformations. For example, for relative velocites along the x^1 axes: $\Lambda^{\nu}_{(1)\nu} = \begin{pmatrix} \gamma & -\beta_1\gamma & 0 & 0 \\ -\beta_1\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

The inverse transformation is:

$$\begin{aligned} dx^{\alpha} &= \frac{\partial x^{\alpha}}{\partial x'^{\mu}} dx'^{\mu} \\ &= \underbrace{\frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x'^{\mu}}{\partial x^{\nu}}}_{\delta^{\alpha}_{\nu}} dx^{\nu} \end{aligned}$$

and it is defined:

$$\begin{aligned} dx^{\alpha} &= \Lambda^{-1}{}^{\alpha}_{\ \mu} dx'^{\mu} \\ dx^{\alpha} &= \underbrace{\Lambda^{-1}{}^{\alpha}_{\ \mu} \Lambda^{\mu}_{\beta}}_{\text{must be } \delta^{\alpha}_{\beta}} dx^{\beta} \end{aligned}$$

The tensorial character of objects is defined by their Lorentz transformation properties. For example

- **rank 0** $\phi = \phi \ \rightarrow$ scalar
- **rank 1** $A'^{\mu} = \Lambda^{\mu}_{\nu} A^{\nu} \ \rightarrow$ contravariant vector, tensor of rank 1
- **rank 2** $A'^{\alpha\beta} = \Lambda^{\alpha}_{\mu}\Lambda^{\beta}_{\nu} A^{\mu\nu} \ \rightarrow$ contravariant tensor of rank 2
- **rank 2** $A'^{\mu}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} A^{\rho}_{\sigma} = \Lambda^{\mu}_{\rho} A^{\rho}_{\sigma} \left(\Lambda^{-1}\right)^{\sigma}_{\nu} \ \rightarrow$ mixed tensor of rank 2
- etc.

From Eq. 21, the 4-gradient gives:

$$\begin{aligned} \frac{\partial A'^{\mu}}{\partial x'^{\rho}} &= \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial A^{\nu}}{\partial x'^{\rho}} + \frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x'^{\rho}} A^{\nu} \\ &= \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\alpha}}{\partial x'^{\rho}} \frac{\partial A^{\nu}}{\partial x^{\alpha}} + \frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial x'^{\rho}} A^{\nu} \end{aligned}$$

We'll deal in only linear transformations for which the last term is zero. If the last term is not zero, then the gradient of A' is not a tensor and the affine connection in non- Euclidean geometries comes into play.

2.2 Standard Tensors

Useful are the following tensors:

2.2.1 The Metric

This is $g_{\mu\nu} = g^{\mu\nu}$ and is useful to raise and lower indices:

$$T^{\rho}_{\mu\rho} = g_{\mu\nu} T^{\nu\sigma}_{\rho} \tag{25}$$

$g_{\alpha\beta}$ and $g^{\alpha\beta}$ are reciprocals, which can easily be shown.

2.2.2 The 4-Gradient Operator

This quantity, while a rank-1 tensor, is not a 4-vector (because it does not transform according to the definition of a 4-vector). Watch the placement of the indices, which influences the odd nature of the sign of the space piece:

$$\frac{\partial}{\partial x^{\mu}} \equiv \partial_{\mu} = \left[\partial/\partial t, +\vec{\nabla}\right] \tag{26}$$

$$\frac{\partial}{\partial x_{\mu}} \equiv \partial^{\mu} = \left[\partial/\partial t, -\vec{\nabla}\right] \tag{27}$$

The, the 4-divergence is properly defined:

$$\frac{\partial A_{\mu}}{\partial x^{\mu}} = \partial^{\mu} A_{\mu} = \frac{\partial A_0}{\partial t} + (-\vec{\nabla}) \cdot (-\vec{A}) \tag{28}$$

$$= \frac{\partial A_0}{\partial t} + (\vec{\nabla} \cdot \vec{A}) \tag{29}$$

which is what one expects.

2.2.3 The Invariant D'Alembertian Operator

This is the double-derivative, or "box" operator:

$$\frac{\partial}{\partial x^{\mu}} \frac{\partial}{x_{\mu}} \equiv \partial_{\mu}\partial^{\mu} \equiv \Box = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \tag{30}$$

2.2.4 The Antisymmetric Tensors

There are different conventions for these, so one has to check.

$$\epsilon_{123} = -\epsilon^{123} = -1 \tag{31}$$

$$\epsilon_{0123} = -\epsilon^{0123} \tag{32}$$

$$\epsilon_{\mu\nu\alpha\beta} = -\epsilon_{\nu\mu\alpha\beta} \tag{33}$$

The latter is called the Levi-Cevita Tensor. The completely antisymmetric nature of these is in the permutation of the indices as in the last line. This means that any contraction with a symmetric tensor gives zero, e.g. $\epsilon_{\mu\nu\alpha\beta} g^{\alpha\beta} = 0$. Some useful contractions of Levi-Cevita Tensors are:

$$\epsilon^{\mu\nu\sigma\rho}\epsilon_{\mu\nu\gamma\delta} = -2d^{\sigma}_{\gamma}\delta^{\rho}_{\delta} + 2\delta^{\rho}_{\gamma}\delta^{\sigma}_{\gamma} \tag{34}$$

$$\epsilon^{\mu\nu\sigma\rho}\epsilon_{\mu\nu\sigma\delta} = -3!\delta^{\rho}_{\gamma} \tag{35}$$

$$\epsilon^{\mu\nu\alpha\rho}\epsilon_{\mu\nu\sigma\rho} = -4! \tag{36}$$

3 Dirac Equation

The Dirac Equation is an operator equation of the standard form:

$$H\psi = \frac{i}{\hbar}\frac{\partial\psi}{\partial t} \tag{37}$$

$$-i\hbar c\vec{\alpha}\cdot\vec{\nabla}\psi + \beta mc^2\psi = \frac{i}{\hbar}\frac{\partial\psi}{\partial t} \tag{38}$$

Here, the Dirac matrices are:

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{39}$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \tag{40}$$

Of course, the matrix in Eq. 40 is short hand for a 4×4 as the Pauli Matrices are implied here are;

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{41}$$

So, in this representation[1], the ψ are 4-component spinors. As usual, the Hamiltonian operator, $H = -i\hbar c\vec{\alpha}\cdot\vec{\nabla} + \beta mc^2$ proscribes what operators represent constants of the motion and explicit calculation shows that the following are true:

$$[H, H] = 0 \tag{42}$$

$$\left[H, p^i\right] = 0 \tag{43}$$

$$i\left[H, \vec{p}\cdot\vec{S}\right] = 0 \tag{44}$$

where

$$\vec{S} \equiv \tfrac{1}{2}\vec{\Sigma} \tag{45}$$

$$\vec{\Sigma} \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{46}$$

Notice that Eq. 44 says that helicity is a constant of the motion, and hence a good relativistic quantum number for a fermion. Neither spin, nor angular momentum are, as can be shown explicitly:

$$[H, S_i] \neq 0 \tag{47}$$

$$[H, L_i] \neq 0 \tag{48}$$

The free-particle solutions will be chosen to be simultaneous eigenfunctions of both H and p.

The covariant form of the Dirac Equation comes from manipulating β and α to produce the "Dirac gamma matrices:"

$$\gamma^0 \equiv \beta \tag{49}$$

$$\vec{\gamma} \equiv \beta\vec{\alpha} \tag{50}$$

$$\gamma^\mu = \left[\gamma^0, \vec{\gamma}\right] \tag{51}$$

[1] This is the minimum representation in order to satisfy Dirac's requirement that the equation be of first order in spatial derivatives. It is not unique.

which satisfy the following *anti*commutation properties:

$$\left.\begin{array}{l}\{\gamma_0,\gamma_i\}=0\\ \{\gamma_i,\gamma_j\}=-2\delta_{ij}\end{array}\right\}\Rightarrow\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu}\mathbb{I} \tag{52}$$

(Here, $\mathbb{I}$ signifies the 4×4 unit matrix.) Particularly useful is the "slash" shorthand for the contraction of a 4-vector with a Dirac gamma matrix.

$$A^\mu\gamma_\mu\equiv \not{A} \tag{53}$$

With these definitions, the covariant Dirac Equation becomes:

$$(i\hbar\gamma^\mu\partial_\mu-mc)\psi(x) = 0 \tag{54}$$

$$(i\not{\partial}-m)\psi = 0 \tag{55}$$

where the second equation is explicitly in "blackboard units" and exhibits the slash notation for the 4-gradient symbol. In momentum space, this is

$$(\gamma^\mu p_\mu-m)\,\psi=0 \tag{56}$$

3.1 Hermiticity Properties of Dirac Matrices

The useful relations involving conjugations are:

$$\beta^\dagger = \beta,\ \text{so } (\gamma^0)^\dagger=\gamma^0 \text{ Hermitian} \tag{57}$$

$$\gamma_i^\dagger = (\beta\alpha_i)^\dagger=\alpha^\dagger\beta^\dagger \tag{58}$$

$$= \alpha_i^\dagger\beta=\alpha_i\beta \tag{59}$$

$$\gamma_i^\dagger = -\beta\alpha_i=-\gamma_i \text{ antiHermitian} \tag{60}$$

$$\text{also} \tag{61}$$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{62}$$

3.2 Free Particle Solutions

Another convention is

$$\bar\psi\equiv\psi^\dagger\gamma^0 \tag{63}$$

which allows for the construction of the conjugate (matrix) equation:

$$\bar\psi\left(i\gamma^\mu\overleftarrow{\partial}_\mu+m\right) = 0 \tag{64}$$

$$\bar\psi\left(\gamma^\mu p_\mu+m\right) = 0 \tag{65}$$

So, collecting all of the forms:

$$(i\not{\partial}-m)\psi(x) = 0 \tag{66}$$

$$\bar\psi(x)\left(i\overleftarrow{\not{\partial}}_\mu+m\right) = 0 \tag{67}$$

$$(\not{p}_\mu-m)\,\psi(p) = 0 \tag{68}$$

$$\bar\psi(p)\left(\overleftarrow{\not{p}}+m\right) = 0 \tag{69}$$

Completing the general set, the Dirac Spinor is a 4-component object:

$$\psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{70}$$

3.2.1 Klein Gordon Connection

Remembering that the Klein Gordon equation is just a statement of the relativistic energy relation, as an operator, then the Dirac Equation must be related. Take the Dirac Equation, Eq. 66 and operate from the left by $i\gamma^\nu\gamma_\nu$ to get

$$(-\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu - i\gamma^\nu\partial_\nu m)\,\psi = 0 \tag{71}$$
$$\left(-\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu - m^2\right)\psi = 0 \tag{72}$$

where the Dirac Equation itself was used in the second term. We can write this second equation again, just changing the dummy summed indices. Then add the two equations together:

$$(-\gamma^\nu\gamma^\mu \underbrace{\partial_\nu\partial_\mu}_{\partial_\mu\partial_\nu} - m^2 - \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2)\psi = 0 \tag{73}$$
$$[-\underbrace{(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu)}_{\{\gamma^\nu,\gamma^\mu\}}\partial_\mu\partial_\nu - 2m^2]\,\psi = 0 \tag{74}$$
$$\left(-2\gamma^{\nu\mu}\partial_\mu\partial_\nu\mathbb{I} - 2m^2\mathbb{I}\right)\psi = 0 \tag{75}$$

where the Dirac-space matrices are now all diagonal, signified by the unit matrix, $\mathbb{I}$. This means that there is a separate equation for every component of ψ. With trivial simplification,

$$\left(\partial^\mu\partial_\nu + m^2\right)\psi_a = 0 \tag{76}$$
$$\left(\Box + m^2\right)\psi_a = 0 \tag{77}$$

So, each component of the Dirac spinor separately satisfies a Klein Gordon equation. This both demonstrates the satisfaction of the relativistic momentum-energy relation, but also points out that the Dirac spinors satisfy a wave equation, so solutions in waves can be readily assumed. With that, we can then postulate generally,

$$\psi_i(x) = Nu_i(p)e^{-ip\cdot x} \tag{78}$$
$$= Nu_i(p)e^{-ip_0x_0+i\vec{p}\cdot\vec{x}} \tag{79}$$

So, the proliferation of indices and vector spaces begins...

This Fourier splitting of the momentum and space pieces is familiar. The $u's$ are typically segregated into the "upper" and the "lower" components, u_A and u_B, with a useful and standard shorthand of a 2 d spinor space representing the 4 d spinor space:

$$u(p) = \begin{pmatrix} u_A(p) \\ u_B(p) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \tag{80}$$

By putting this back into the Dirac Equation using the 2 d form, one can write a Schrodinger-like eigenvalue equation using the Hamiltonian $H = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}$ which is of course a 4x4 matrix.

$$\begin{pmatrix} (p_0 - m) & -\vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & (p_0 + m) \end{pmatrix}\begin{pmatrix} u_A(p) \\ u_B(p) \end{pmatrix} = 0 \tag{81}$$

Decoupling these equations into two 2x2 matrix equations involves setting the determinant to zero, which results in a familiar constraint:

$$
\begin{aligned}
(p_0 - m)9p_0 + m) - (-\vec{\sigma}\cdot\vec{p})(-\vec{\sigma}\cdot\vec{p}) &= 0 && (82)\\
p_0^2 - m^2 - \underbrace{\vec{\sigma}\cdot\vec{p}\vec{\sigma}\cdot\vec{p}}_{\vec{p}\cdot\vec{p}} &= 0 && (83)\\
p_0^2 - m^2 - \vec{p}^{\,2} &= 0 \quad \text{or} && (84)\\
p_0 &= \vec{p}^{\,2} + m^2 && (85)
\end{aligned}
$$

For a given momentum, the energy values will be $p_0 = \pm\sqrt{\vec{p}^{\,2} + m^2} = \pm E$, where E is a number and always, $E > 0$.

If we choose here $p_0 = +E$, then the coupled equations are

$$
\begin{pmatrix} (E-m) & -\vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & (E+m) \end{pmatrix}\begin{pmatrix} u_A(p) \\ u_B(p) \end{pmatrix} = 0 \qquad (86)
$$

which leads to

$$
\begin{aligned}
u_A &= \frac{\vec{\sigma}\cdot\vec{p}}{E-m}u_B && (87)\\
u_B &= \frac{\vec{\sigma}\cdot\vec{p}}{E+m}u_A \quad \text{leading to} && (88)\\
\left(E^2 - m^2\right)u_A &= \vec{p}^{\,2}u_A && (89)
\end{aligned}
$$

If we choose here $p_0 = -E$, then the same steps give an identical equation. Within the A and the B representations, the components of $u_{A,B}$ are arbitrary. For the states which are plane waves and eigenstates of just energy/momentum, the convention is to choose $u_1 =$ and $u_2 = 0$ or for the other solution, $u_1 = 0$ and $u_2 = 1$. The same thing holds for u_B's states. Then, for the positive energy solution:

$\boxed{p_0 = +E}$

$$
u(p)_+ = Nu(p) = N\begin{pmatrix} u_A \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}u_A \end{pmatrix} \qquad (90)
$$

For example, solution 1 is explicitly:

$$
\begin{aligned}
u(p)_+^{(1)} &= N\begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{E+m} \\ \frac{p_1+ip_2}{E+m} \end{pmatrix} \quad \text{Solutions 1\&2 are summarized as} && (91)\\
u(p)_+^{(1)} &= N\begin{pmatrix} \chi^{1,2} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{1,2} \end{pmatrix} && (92)
\end{aligned}
$$

Where

$$
\begin{aligned}
\chi^1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} && (93)\\
\chi^2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} && (94)
\end{aligned}
$$

Likewise, for $\boxed{p_0 = -E}$

$$
u(p)_- = Nu(p) = N\begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}u_B \\ u_B \end{pmatrix} \qquad (95)
$$

Solutions 1&2 are summarized as

$$u(p)_-^{(3,4)} = N \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{3,4} \\ \chi^{3,4} \end{pmatrix} \tag{96}$$

Where

$$\chi^3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{97}$$

$$\chi^4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{98}$$

3.2.2 Normalization

There are two conventions for spinor normalization. Both start from the general statement:

$$2p^\mu \bar{u}(p)u(p) = 2mu^\dagger u \quad \text{which for } \mu = 0 \text{ gives} \tag{99}$$

$$u^\dagger u = \frac{p^0}{m}\bar{u}(p)u(p) \tag{100}$$

The two conventions are

$$1 : u^\dagger u = \gamma = \frac{p^0}{m} \Rightarrow \bar{u}(p)u(p) = 1 \tag{101}$$

$$2 : u^\dagger u = 2m\gamma = 2p^0 \Rightarrow \bar{u}(p)u(p) = 2m \tag{102}$$

We'll use convention 2, as it's kinder to massless fermions. So, then for the two energy conditions:

$$p^0 = +E : \quad \bar{u}_i(p)u_j(p) = 2m\delta_{ij} \quad i,j = 1,2 \tag{103}$$

$$p^0 = -E : \quad \bar{u}_i(p)u_j(p) = -2m\delta_{ij} \tag{104}$$

With this selection, from terms like Eq. 92, the overall spinor normalization can be determined. For our choice, this turns out to be $N = \sqrt{E+m}$. With the definition $p_\pm \equiv p_1 \pm ip_2$, the explicit spinors as eigenfunctions of energy and momentum are:

$$\psi_1(x) = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{E+m} \\ \frac{p_+}{E+m} \end{pmatrix} e^{-iEt+i\vec{p}\cdot\vec{x}} \tag{105}$$

$$\psi_2(x) = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_-}{E+m} \\ \frac{-p_3}{E+m} \end{pmatrix} e^{-iEt+i\vec{p}\cdot\vec{x}} \tag{106}$$

$$\psi_3(x) = \sqrt{E+m}\begin{pmatrix} \frac{-p_3}{E+m} \\ \frac{-p_+}{E+m} \\ 1 \\ 0 \end{pmatrix} e^{iEt+i\vec{p}\cdot\vec{x}} \tag{107}$$

$$\psi_4(x) = \sqrt{E+m}\begin{pmatrix} \frac{-p_-}{E+m} \\ \frac{p_3}{E+m} \\ 0 \\ 1 \end{pmatrix} e^{iEt+i\vec{p}\cdot\vec{x}} \tag{108}$$

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3.2.3 Negative Energy Solutions

By an analysis of the charge conjugation properties of the Dirac spinor solutions and the simple (algebraic) observation that

$$\underbrace{e^{-ip_0t}}_{\text{an electron wavefunction for }+\text{E}} = \underbrace{e^{-i(-p_0)(-t)}}_{\text{an electron wavefunction for -E and backwards in time}} , \quad (109)$$

the charge conjugate states, $\psi_{\mathcal{C}i}$ can be related to the negative energy states, $3, 4$:

$$\begin{aligned}
\psi_{\mathcal{C}1}(\vec{p}) &= -\psi_4(-\vec{p}) \text{ and } E > 0 \text{ for } e^+ && (110)\\
\psi_{\mathcal{C}2}(\vec{p}) &= +\psi_3(-\vec{p}) \text{ and } E > 0 \text{ for } e^+ && (111)\\
\psi_{\mathcal{C}3}(\vec{p}) &= +\psi_2(-\vec{p}) \text{ and } E < 0 \text{ for } e^+ && (112)\\
\psi_{\mathcal{C}4}(\vec{p}) &= -\psi_1(-\vec{p}) \text{ and } E < 0 \text{ for } e^+ && (113)
\end{aligned}$$

So, the negative energy solutions can be gotten rid of altogether by choosing the physical states to be $\psi_1, \psi_2, \psi_{\mathcal{C}1}, \psi_{\mathcal{C}2}$. This leads to the familiar definitions for the momentum space spinors for physical electron and positron (fermion and antifermion) states:

$$\begin{aligned}
u_1(p) &= \text{electron with } E > 0 \text{ and "spin" } = 1/2 && (114)\\
u_2(p) &= \text{electron with } E > 0 \text{ and "spin" } = \text{-}1/2 && (115)\\
v_1(p) \equiv u_{\mathcal{C}1}(p) &= \text{positron with } E > 0 \text{ and "spin" } = 1/2 && (116)\\
v_2(p) \equiv u_{\mathcal{C}2}(p) &= \text{positron with } E > 0 \text{ and "spin" } = \text{-}1/2 && (117)
\end{aligned}$$

The $v's$ satisfy a Dirac Equation, but for the opposite p^μ. So, here then is the summary of the relevant physical relationships:

$$\begin{aligned}
(\not{p} - m)\, u_i(p) &= 0 && (118)\\
(\not{p} + m)\, v_i(p) &= 0 && (119)\\
\bar{u}(p)\, (\not{p} - m) &= 0 && (120)\\
\bar{v}(p)\, (\not{p} + m) &= 0 && (121)\\
\bar{u}_i(p) u_j(p) &= 2m\delta_{ij} && (122)\\
u_i^\dagger u_j &= 2E\delta_{ij} && (123)\\
\bar{v}_i(p) v_j(p) &= -2m\delta_{ij} && (124)\\
v_i^\dagger v_j &= 2E\delta_{ij} && (125)\\
\bar{v}u &= \bar{u}v = 0 && (126)
\end{aligned}$$

$$\begin{aligned}
u_s(p) &= \sqrt{E+m} \begin{pmatrix} \chi^s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^s \text{ where } i = 1,2 \end{pmatrix} && (127)\\
v_s(p) &= \sqrt{E+m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi^{-s} \\ \chi^{-s} \end{pmatrix} \text{ where } i = 1,2\text{and} && (128)\\
\chi^1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +1/2 \text{ in rest frame} && (129)\\
\chi^2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1/2 \text{ ''} && (130)\\
\chi^{-1} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 11/2 \text{ in rest frame} && (131)\\
\chi^{-2} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} = +1/2 \text{ ''} && (132)
\end{aligned}$$

The important thing to notice about this choice: these are **not** eigenstates of helicity, only of energy and momentum. The simple "Pauli-like" spinor for the "spin" part is a particular choice, not a general one. That is, a direction in space has already been chosen.

Completeness is important. The original 4 spinors, positive and negative energy states, form a complete set:

$$\sum_{j=1}^{4} u^{(j)}(p)_i u^{(j)}(p)^{\dagger}_m = 2p^0 \delta_{im} \mathbb{I} \tag{133}$$

Notice all of the different vector spaces in this equation: the spinor components are represented by the j index. This is a matrix equation—an outer product— and each matrix element of that 4×4 is represented...these are the i, m indices. FInally, there is a spacetime index, namely the 0 in the four momentum p. With the positron states, Completeness takes on a different form:

$$\sum_{j=1}^{2} u^{(j)}(p)_i \bar{u}^{(j)}(p)_m - \sum_{j=1}^{2} v^{(j)}(p)_i \bar{v}^{(j)}(p)_m = \delta_{im} 2m \tag{134}$$

3.2.4 Cleanup

Finally, remember that there are useful Diracology terms. Arbitrarily:

$$\sigma^{\mu\nu} \equiv i/2\,[\gamma^\mu, \gamma^\nu] \tag{135}$$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{136}$$

$$\gamma^{5\dagger} = \gamma^5 \tag{137}$$

$$\gamma^5\gamma^5 = \mathbb{I} \tag{138}$$

$$\left\{\gamma^5, \gamma^\mu\right\} = 0 \tag{139}$$