TAKING STOCK:

We defined the Time Ordered Product (or Wick Time Ordered Product) as

$$T[\Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)] = (-1)^P\,\Phi(x_i)\Phi(x_j)\cdots\Phi(x_n)$$

where

P = # fermion operator permutations

$x_{i_0} > x_{j_0} > \cdots x_{n_0}$

We found that for 2 operators

$$T[A(x)B(y)] = \,:A(x)B(y): + \langle 0|T(A(x)B(y))|0\rangle$$

.. true for either $x_0 > y_0$ or $y_0 > x_0$

.. true for either fermions or bosons

We defined the <u>number</u>

$$\langle 0|T(AB)|0\rangle \equiv \underbracket{A(x)B(y)}$$ called the "contraction of A & B"

I didn't prove, but included a demonstration for 3 operators and then implied by induction the validity of Wick's Theorem

$$\begin{aligned} T(ABC\ldots XYZ) &= \,:ABC\ldots XYZ: \\ &+ :\underbracket{AB}C\ldots XYZ: + \text{other } \underbracket{\;\;} \\ &+ :\underbracket{AB}C\ldots \underbracket{XY}Z: + \text{other } \underbracket{\;\;}\,\underbracket{\;\;} \\ &\;\vdots \end{aligned}$$

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until all terms are contracted.

Wick's Theorem is a labor-saving device — and led to a heuristic and intuitive picture of the physics (Feynman diagrams) — but it is not necessary in order to do calculations. For example, Sakurai doesn't use it.

So, in order to do a calculation to 2nd order one needs the S matrix operator

$$S^{(2)} = \frac{(-i)^2}{2} \int_{-\infty}^{\infty} d^4x_1 \int_{-\infty}^{\infty} d^4x_2 \; P\left[\mathcal{L}_I(x_1)\, \mathcal{L}_I(x_2) \right]$$

from which one creates the $\mathcal{T}$ matrix element

$$\mathcal{T}^{(2)} = i \langle \text{final states} \,|\, S^{(2)} \,|\, \text{initial states} \rangle$$

We showed that for $\mathcal{L}_I$'s with pairs of fermion-antifermion operators that $P[\;] = T[\;]$ so,

$$S^{(2)} = \frac{(-i)^2}{2} \int_{-\infty}^{\infty} d^4x_1 \int_{-\infty}^{\infty} d^4x_2 \; T\left[\mathcal{L}_I(x_1)\, \mathcal{L}_I(x_2) \right]$$

For the 2nd order interaction between charged fermions and the electromagnetic field,

$$\mathcal{L}_I(x) = -e\bar{\Psi}(x)\gamma^\mu \Psi(x) A_\mu(x)$$

and

$$S^{(2)} = \frac{(-ie)^2}{2} \int_{-\infty}^{\infty} d^4x_1 \int_{-\infty}^{\infty} d^4x_2 \; T\left[\bar{\Psi}(x_1)\gamma^\mu \Psi(x_1) A_\mu(x_1)\, \bar{\Psi}(x_2)\gamma^\nu \Psi(x_2) A_\nu(x_2) \right]$$

In general

$$T[\quad] = \gamma^{\mu}_{ij}\gamma^{\nu}_{\ell m}\, T\left[\bar{\Psi}_i(x_1)\Psi_j(x_1)A_\mu(x_1)\bar{\Psi}_\ell(x_2)\Psi_m(x_2)A_\nu(x_2)\right]$$

which from Wick's Theorem becomes.

$$= \gamma^{\mu}_{ij}\gamma^{\nu}_{\ell m} \sum_{W} \mathcal{O}^{(w)}_{\mu\nu}$$

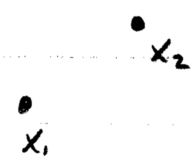
59 terms .. 3 of which are nonzero.

— notice: no specific process, just general electron-photon interaction (which includes positron-photon).

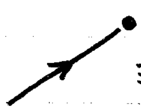
The individual terms in the sum $O^{(1)} - O^{(8)}$ each represent potential 2^{nd} order interactions among 2 fermions and a photon. The Feynman graph techniques (Feynman 1949) are ways of organizing the calculations in an intuitive, pictoral way.

I'll develop things first in configuration space and do a calculation. But, we'll quickly find that momentum space rules are more useful and it will become rote at that point.

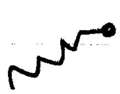
So, for a 2^{nd} order calculation we have 2 spacetime vertices



For an unpaired (uncontracted) operator $\psi(x)$ or $\overline{\psi}(x)$, we draw a line with an arrow to or from a vertex

 $\equiv \psi(x)$  $\equiv \overline{\psi}(x)$

An unpaired $A_\mu(x)$ is either (no arrow)

  $\equiv A_\mu(x)$

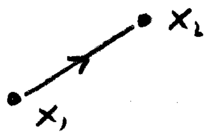
The contracted terms, remember, are

$$T\langle 0|\,\alpha(x_1)\,\beta(x_2)\,|0\rangle \equiv \underbrace{\alpha(x_1)\,\beta(x_2)}$$

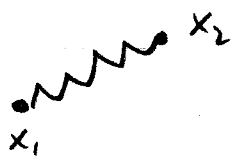
but only $\quad T\langle 0|\,\bar{\Psi}(x_1)\,\Psi(x_2)\,|0\rangle = \underbrace{\bar{\Psi}(x_1)\,\Psi(x_2)}$

and $\quad T\langle 0|\,A_\mu(x_1)\,A_\nu(x_2)\,|0\rangle = \underbrace{A(x_1)\,A(x_2)}$

are non-zero. They are represented as connectors between space time points:



$$\equiv \langle 0|\,\bar{\Psi}(x_1)\,\Psi(x_2)\,|0\rangle = \underbrace{\bar{\Psi}(x_1)\,\Psi(x_2)}$$



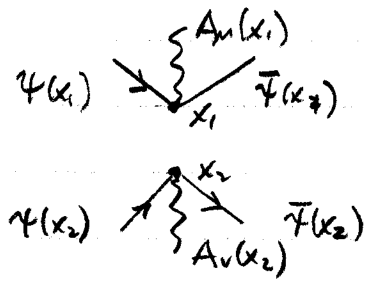
$$\equiv \langle 0|\,A_\mu(x_1)\,A_\nu(x_2)\,|0\rangle = \underbrace{A_\mu(x_1)\,A_\nu(x_2)}$$

Now we can break down the 8 non-zero terms in the Dyson expansion and sketch a space time picture for each one -- recognizing potential physics processes as we go.

$$\boxed{\Theta^{(1)}_{\mu\nu}} = :\bar{\Psi}_i(x_1)\Psi_j(x_1)A_\mu(x_1)\,\bar{\Psi}_k(x_2)\Psi_m(x_2)A_\nu(x_2):$$

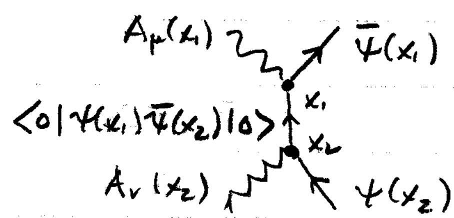
– the fully normal-ordered term -- no contractions,

$\Rightarrow$ no connection between x_1 and x_2



$$\boxed{\Theta^{(2)}_{\mu\nu}} = :\bar{\Psi}_i(x_1)\underbrace{\Psi_j(x_1)A_\mu(x_1)\bar{\Psi}_k(x_2)}\Psi_m(x_2)A_\nu(x_2):$$

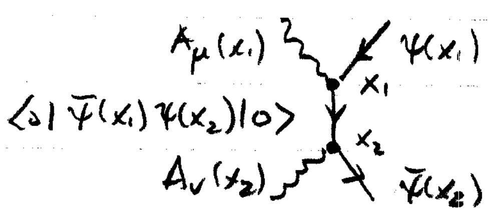
$$= \underbrace{\Psi_j(x_1)\bar{\Psi}_k(x_2)} : \underbrace{\bar{\Psi}_i(x_1)A_\mu(x_1)}_{x_1}\underbrace{\Psi_m(x_2)A_\nu(x_2)}_{x_2}:$$



compton scattering.

$$\boxed{\Theta^{(3)}_{\mu\nu}} = :\underbrace{\bar{\Psi}_i(x_1)\Psi_j(x_1)A_\mu(x_1)\bar{\Psi}_k(x_2)\Psi_m(x_2)}A_\nu(x_2):$$

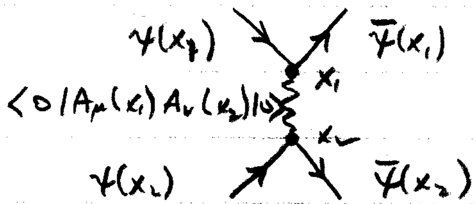
$$= \underbrace{\bar{\Psi}_i(x_1)\Psi_m(x_2)} : \Psi_j(x_1)A_\mu(x_1)\bar{\Psi}_k(x_2)A_\nu(x_2):$$



which is topologically indistinguishable from $\Theta^{(2)}$.

$$\boxed{\Theta^{(4)}_{\mu\nu}} = \,:\bar{\Psi}(x_1)\Psi(x_1)A_\mu(x_1)\bar{\Psi}(x_2)\Psi(x_2)A_\nu(x_2):$$

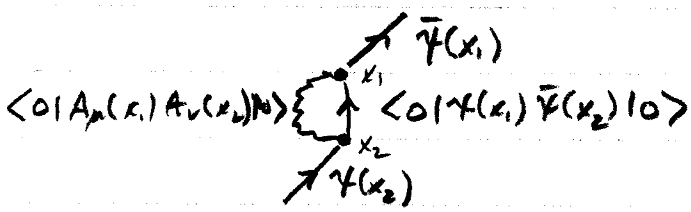
$$= A_\mu(x_1)A_\nu(x_2)\,:\bar{\Psi}(x_1)\Psi(x_1)\bar{\Psi}(x_2)\Psi(x_2):$$



Moller Scattering,
Bhabha Scattering
Positronium

$$\boxed{\Theta^{(5)}_{\mu\nu}} = \,:\bar{\Psi}(x_1)\Psi(x_1)A_\mu(x_1)\bar{\Psi}(x_2)\Psi(x_2)A_\nu(x_2):$$

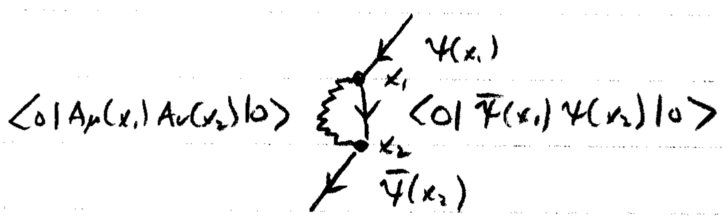
$$= \Psi(x_1)\bar{\Psi}(x_2)A_\mu(x_1)A_\nu(x_2)\,:\bar{\Psi}(x_1)\Psi(x_2):$$



self-energy

$$\boxed{\Theta^{(6)}_{\mu\nu}} = \,:\bar{\Psi}(x_1)\Psi(x_1)A_\mu(x_1)\bar{\Psi}(x_2)\Psi(x_2)A_\nu(x_2):$$

$$= \bar{\Psi}(x_1)\Psi(x_2)A_\mu(x_1)A_\nu(x_2)\,:\Psi(x_1)\bar{\Psi}(x_2):$$

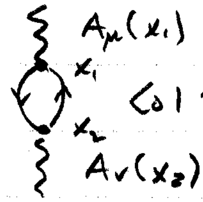


topologically
indistinguishable
from $\Theta^{(5)}$

$$\boxed{\Theta^{(7)}_{\mu\nu}} = :\bar\Psi(x_1)\Psi(x_1)A_\mu(x_1)\,\bar\Psi(x_2)\Psi(x_2)A_\nu(x_2):$$

$$= \bar\Psi(x_1)\Psi(x_2)\,\Psi(x_1)\bar\Psi(x_2)\;:A_\mu(x_1)A_\nu(x_1):$$

$\langle 0|\bar\Psi(x_1)\Psi(x_2)|0\rangle$



$\langle 0|\Psi(x_1)\bar\Psi(x_2)|0\rangle$

vacuum polarization

$$\boxed{\Theta^{(8)}_{\mu\nu}} = :\bar\Psi(x_1)\Psi(x_1)A_\mu(x_1)\,\bar\Psi(x_2)\Psi(x_2)A_\mu(x_2):$$

$$= \bar\Psi(x_1)\Psi(x_2)\,\Psi(x_1)\bar\Psi(x_2)\,A_\mu(x_1)A_\nu(x_2)$$

$\langle 0|\bar\Psi(x_1)\Psi(x_2)|0\rangle$

$\langle 0|\Psi(x_1)\bar\Psi(x_2)|0\rangle$

$\langle 0|A_\mu(x_1)A_\nu(x_2)|0\rangle$

vacuum fluctuations

This is general. Let's look at $\theta^{(2)}$ —

$$\theta^{(2)}_{\mu\nu} = \langle 0 | T[\psi_j(x_1)\bar{\psi}_k(x_2)] | 0 \rangle : \bar{\psi}_i(x_1) A_\mu(x_1) \psi_m(x_2) A_\nu(x_2) :$$

$$\downarrow$$

$$: \bar{\psi}_i(x_1) \psi_m(x_2) A_\mu(x_1) A_\nu(x_2) :$$

$$\parallel$$

$$\underbrace{: \bar{\psi}_i(x_1) \psi_m(x_2) :}\, : A_\mu(x_1) A_\nu(x_2) :$$

$$:(\overset{a^+}{\bar{\psi}^-} + \overset{b}{\bar{\psi}^+})(\overset{a}{\psi^+} + \overset{b^+}{\psi^-}):$$

$$= \; :\bar{\psi}^-\psi^+: + :\bar{\psi}^-\psi^-: + :\bar{\psi}^+\psi^+: + :\bar{\psi}^+\psi^-:$$

$$\propto \; :a^+a: + :a^+b^+: + :ba: + :bb^+:$$

$$= \; \underset{①}{a^+a} + \underset{②}{a^+b^+} + \underset{③}{ba} - \underset{④}{b^+b}$$

Where we go from here depends on the actual process.

Suppose we are considering classical Compton scattering: $e\gamma \to e\gamma$.

then $\quad |\text{initial}\rangle = |e^-, \gamma\rangle$

$\qquad\;\; |\text{final}\rangle = |e^-, \gamma\rangle$

Look at just the electron Fock space terms..

to distinguish different momentum

$$\langle e'| : \bar{\Psi}(x_1)\Psi(x_2) : |e'\rangle \to \langle 0| a' : \bar{\Psi}(x_1)\Psi(x_2) : a^{+\prime}|0\rangle$$

$\{a, a^+\} \neq 0$
$\{a, a\} = \{a^+ a^+\} =$
$\{b, b\} = \{b^+, b^+\} = 0$
$\{b, b^+\} \neq 0$
$\{a, b\} = 0$

① $\langle 0| a' a^+ a \, a^{+\prime}|0\rangle$
$\neq 0$

② $\langle 0| a' a^+ b^+ a^{+\prime}|0\rangle$
$= -\langle 0| a' b^+ a^+ a^{+\prime}|0\rangle$
$= \langle 0| b^+ a' a^+ a^{+\prime}|0\rangle = 0$

③ $\langle 0| a' b a \, a^{+\prime}|0\rangle$
$= -\langle 0| a' a b a^{+\prime}|0\rangle$
$= \langle 0| a' a a^{+\prime} b|0\rangle = 0$

④ $\langle 0| a' b^+ b a^{+\prime}|0\rangle$
$= -\langle 0| b^+ a' b a^{+}|0\rangle = 0$

So, only $\langle e| \bar{\Psi}^-(x_1)\Psi^+(a_2)|e\rangle$ survives. The same thing is true for the photon

From $\Theta^{(3)}_{\mu\nu}$ we get contributions

$$\Theta^{(3)}_{\mu\nu} = \langle 0| T[\bar{\Psi}_i(x_1)\Psi_m(x_2)]|0\rangle : \underbrace{\Psi_j(x_1)\bar{\Psi}_\ell(x_2)}_{} : : A_\mu(x_1)A_\nu(x_2) :$$

$$\downarrow$$

$$ab + a^+a + b^+b + b^+a^+$$

$$\downarrow \text{ survives again}$$

$$-\bar{\Psi}_\ell^-(x_2)\Psi_j^+(x_1)$$