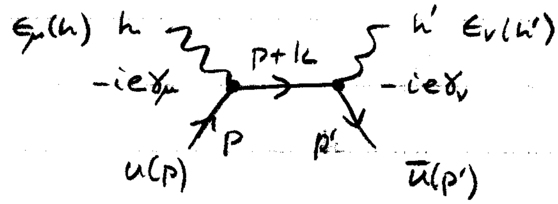
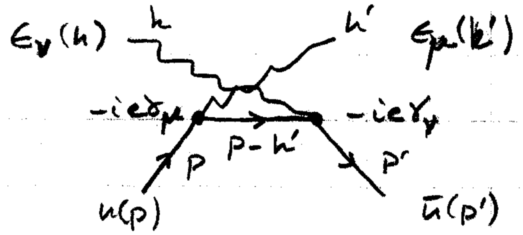
23 Compton-finish1

Now let's do the Compton calculation as if we knew the rules all along. I want the cross section for $\gamma e \to \gamma e$ for an initial electron at rest.

$\gamma(k) + e(p) \to \gamma(k') + e(p')$ to 2^{nd} order.



$$(-i)\,\bar{u}^{(f)}(p')\,\epsilon_{\nu(\lambda')}\,(-ie\gamma^\nu)\,\frac{i}{\not p+\not k-m}\,(-ie\gamma^\mu)\,\epsilon_{\mu(\lambda)}\,u^{(i)}(p)$$



$$(-i)\,\bar{u}^{(f)}(p')\,(-ie\gamma^\mu)\,\epsilon_{\mu(\lambda)}\,\frac{i}{\not p-\not k'-m}\,(-ie\gamma^\nu)\,\epsilon_{\nu(\lambda)}\,u^{(i)}(p)$$

so,

$$T_{fi} = -e^2\,\bar{u}^{(f)}(p')\,\not\epsilon'\,\frac{1}{\not p+\not k-m}\,\not\epsilon\,u^{(i)}(p)$$

$$-e^2\,\bar{u}^{(f)}(p')\,\not\epsilon\,\frac{1}{\not p-\not k'-m}\,\not\epsilon'\,u^{(i)}(p)$$

simplify:

$$\frac{1}{\not p+\not k-m} = \frac{\not p+\not k+m}{(p+k)^2-m^2} = \frac{\not p+\not k+m}{\cancel{p^2}+\cancel{k^2}+2p\cdot k-\cancel{m^2}}$$

$$= \frac{\not p+\not k+m}{2p\cdot k}$$

$$\frac{1}{\not p-\not k'-m} = \frac{\not p-\not k+m}{-2p\cdot k'}$$

so,

$$T_{fi} = -e^2\Big[\ \bar{u}^{(f)}(p')\,\not{\epsilon}'\left(\frac{\not{p}+\not{k}+m}{2p\cdot k}\right)\not{\epsilon}\,u^{(i)}(p).$$

$$-\ \bar{u}^{(f)}(p')\,\not{\epsilon}\left(\frac{\not{p}-\not{k}'+m}{2p\cdot k}\right)\not{\epsilon}'\,u^{(i)}(p)\Big]$$

$$= -e^2\bar{u}^{(f)}(p')\Big[\ \not{\epsilon}'\ \frac{\not{p}+\not{k}+m}{2p\cdot k}\ \not{\epsilon} - \not{\epsilon}\ \frac{\not{p}-\not{k}'+m}{2p\cdot k'}\ \not{\epsilon}'\Big]u^{(i)}(p)$$

continue simplifying –

$$\{(\not{p}+m)\not{\epsilon}\,u^{(i)}(p)\} = (p_\alpha\gamma^\alpha+m)\epsilon_\beta\gamma^\beta u(p)$$

$$= (p_\alpha\epsilon_\beta\gamma^\alpha\gamma^\beta + m\not{\epsilon})u(p)$$

$$= (-p_\alpha\epsilon_\beta\gamma^\beta\gamma^\alpha + p_\alpha\epsilon_\beta 2g^{\alpha\beta} + m\not{\epsilon})u(p)$$

$$= [\not{\epsilon}(-\not{p}+m)u(p) + 2p\cdot\epsilon\, u(p)]$$

from Dirac equation – asymptotically free –

$$(\not{p}-m)u(p) = 0$$

$$\not{p}u = mu \Rightarrow (-\not{p}+m)u(p) = (-\not{p}+\not{p})u(p)$$
$$= 0$$

$$\{\ \} = 2p\cdot\epsilon\ u^{(i)}(p)$$

likewise,

$$(\not{p}+m)\not{\epsilon}'u(p) = 2p\cdot\epsilon' u(p)$$

so,

$$T_{fi} = -e^2\bar{u}^{(f)}(p')\Big[\frac{1}{2p\cdot k}\not{\epsilon}'(2p\cdot\epsilon + \not{k}\not{\epsilon}) - \frac{1}{2p\cdot k'}\not{\epsilon}(2p\cdot\epsilon' - \not{k}'\not{\epsilon}')\Big]u^{(i)}(p)$$

We need to choose a gauge..

Gauge invariance says: $A^\mu(x) \to A^\mu(x) + \partial^\mu\theta(x)$ and $\psi(x) \to \psi(x)\,e^{-i\theta(x)}$ $\Big\}$ invariance

In momentum space.. $\epsilon^\mu(k) \to \epsilon^\mu(k) + \beta k^\mu$

$\epsilon^\mu(k') \to \epsilon^\mu(k') + \beta k^{\mu\prime}$

The above amplitude is invariant with respect to these → neither term alone is. That is, both graphs must be present in order to preserve gauge invariance

I'll choose Coulomb gauge and a frame in which –

$$\begin{aligned} \epsilon^\mu &= (0, \vec{\epsilon}) \\ \epsilon^{\mu\prime} &= (0, \vec{\epsilon}\,') \\ p^\mu &= (m, 0) \end{aligned} \quad \Rightarrow \quad \vec{k}\cdot\vec{\epsilon} = \vec{k}'\cdot\vec{\epsilon}\,' = 0$$

$$p\cdot\epsilon = p\cdot\epsilon' = 0 \qquad \text{4 vectors}$$

So, the amplitude further simplifies.

$$T_{fi} = -e^2\,\bar{u}^{(f)}(p')\left[\frac{1}{2p\cdot k}\,\not{\epsilon}'\,\not{k}\,\not{\epsilon} + \frac{1}{2p\cdot k'}\,\not{\epsilon}\,\not{k}'\,\not{\epsilon}'\right]u^{(i)}(p)$$

The probability and then cross section requires us to square this. One <u>could</u> do this explicitly with spinor components and the Dirac matrices explicitly manipulated. However, there is a much easier way.

Suppose we will have unpolarized photons initially and an unpolarized electron "target". Further, we will presume no attempt to measure the polarization of the outgoing photon or electron — (we'll detect the outgoing photon.) FIRST, deal with electron stuff —

average initial electron spin states $\frac{1}{(2S+1)}$

So

$$\bar{\sum_i} \sum_f |T_{fi}|^2 = e^4 \bar{\sum_i} \sum_f \left| \bar{u}^{(f)}(p') \Gamma u^{(i)}(p) \right|^2$$

sum final electron spin states

all stuff inside [ ] in T_{fi}

now only sum

$$= e^4 \frac{1}{2} \sum_i \sum_f \bar{u}_h^{(f)}(p') \Gamma_{hj} u_j^{(i)}(p) \left[\bar{u}_n^{(f)}(p') \Gamma_{nm} u_m^{(i)}(p) \right]^\dagger$$

keeping track of the Dirac space matrix indices.

look:

$$\begin{aligned} [\ ]^\dagger = (u^\dagger \gamma^0 \Gamma u)^\dagger &= u^\dagger(p) \Gamma^\dagger \gamma^{0\dagger} u(p') \\ &= u^\dagger(p) \gamma^0 \gamma^0 \Gamma^\dagger \gamma^0 u(p') \\ &\equiv \bar{u}(p) \bar{\Gamma} u(p') \end{aligned}$$

where the standard notation $\bar{\Gamma} \equiv \gamma^0 \Gamma^\dagger \gamma^0$

So,

$$\bar{\sum_i} \sum_f |T|^2 = e^4 \cdot \frac{1}{2} \cdot \sum_i \sum_f \bar{u}_h^{(f)}(p') \Gamma_{hj} u_j^{(i)}(p) \bar{u}_m^{(i)}(p) \bar{\Gamma}_{mn} u_n^{(f)}(p')$$

the matrix elements can be rearranged — just numbers.

$$= \frac{e^4}{2} \underbrace{\sum_i u_j^{(i)}(p) \bar{u}_m^{(i)}(p)} \underbrace{\sum_f u_n^{(f)}(p') \bar{u}_h^{(f)}(p')} \Gamma_{hj} \bar{\Gamma}_{mn}$$

completeness → projection operators

Since we're not measuring helicities, we'll only project the + energy electron states.

$$= \frac{e^4}{2} \; (\not{p}+m)_{jm} \, (\not{p}'+m)_{nh} \; \Gamma_{hj} \, \bar{\Gamma}_{mn}$$

arrange numbers back to be proper order for matrix multiplication

$$= \frac{e^4}{2} \; (\not{p}'+m)_{nh} \; \Gamma_{hj} \; (\not{p}+m)_{jm} \; \bar{\Gamma}_{mn}$$

Casimir's Trick.

remember, a number. – in fact, the Trace

$$= \frac{e^4}{2} \, \mathrm{Tr}\left[\, (\not{p}'+m) \, \Gamma \, (\not{p}+m) \, \bar{\Gamma} \right]$$

For this problem, $\Gamma = \frac{1}{2p\cdot k} \not{\epsilon}' \not{k} \not{\epsilon} + \frac{1}{2p\cdot k'} \not{\epsilon} \not{k}' \not{\epsilon}'$

So, we need to take the Trace of Dirac matrices. lots of useful formulae and theorems:

- $\mathrm{Tr}(\gamma^\mu)$: $\mu = 0 \Rightarrow \mathrm{Tr}(\gamma^0) = 0$
  $\mu = i \Rightarrow \mathrm{Tr}(\gamma^i) = \mathrm{Tr}\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} = 0$

so $\boxed{\mathrm{Tr}(\gamma^\mu) = 0}$

not very exciting.

- $\mathrm{Tr}(\gamma^\mu \gamma^\nu)$: this is useful for $\mathrm{Tr}(\not{a}\not{b}) = a_\mu b_\nu \mathrm{Tr}(\gamma^\mu\gamma^\nu)$

problem

note $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) + \mathrm{Tr}(\gamma^\nu\gamma^\mu) = 2g^{\mu\nu}\,\mathrm{Tr}(\mathbb{1})$$

since $\mathrm{Tr}(abc) = \mathrm{Tr}(cab) = \cdots$ this becomes,

$$2\,\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 2g^{\mu\nu}\cdot 4$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

so

$$\boxed{\mathrm{Tr}(\not{a}\not{b}) = 4a_\mu b_\nu g^{\mu\nu} = 4a\cdot b}$$

- $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda)$: trick... note $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$$\begin{aligned}
\gamma^5\gamma^5 &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= +\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = \gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\
&= \gamma^1\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3 \\
&= -\gamma^2\gamma^3\gamma^2\gamma^3 = +\gamma^2\gamma^2\gamma^3\gamma^3 = 1
\end{aligned}$$

so,

$$\begin{aligned}
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda) &= \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^5\gamma^5) \\
&= \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^5)
\end{aligned}$$

also $= -\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^5\gamma^5) = -\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda)$!

so,

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda) = 0$$

indeed, $\boxed{\mathrm{Tr}(\text{odd number } \gamma\text{'s}) = 0}$

- For an even number, there's a theorem:

$$\mathrm{Tr}(\not{a}_1 \not{a}_2 \not{a}_3 \cdots \not{a}_n) = a_1 \cdot a_2 \, \mathrm{Tr}(\not{a}_3 \cdots \not{a}_n) - a_1 \cdot a_3 \, \mathrm{Tr}(\not{a}_2 \not{a}_4 \cdots \not{a}_n) + \cdots + (a_1 \cdot a_n)\, \mathrm{Tr}(\not{a}_2 \not{a}_3 \cdots \not{a}_n)$$

this is since $\quad \not{a}_1 \not{a}_2 = -\not{a}_2 \not{a}_1 + 2 a_1 \cdot a_2$

so,

$$\mathrm{Tr}(\not{a}_1 \not{a}_2 \not{a}_3 \cdots \not{a}_n) = -\mathrm{Tr}(\not{a}_2 \not{a}_1 \cdots \not{a}_n) + 2 a_1 \cdot a_2 \, \mathrm{Tr}(\not{a}_3 \cdots \not{a}_n)$$

which one just continues. –

- $n = 4$ happens quite a bit. –

$$\boxed{\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4\left[(a_1 \cdot a_2)(a_3 \cdot a_4) - (a_1 \cdot a_3)(a_2 \cdot a_4) + (a_1 \cdot a_4)(a_2 \cdot a_3) \right]}$$

- Finally, $\boxed{\mathrm{Tr}(\gamma_5 \not{a} \not{b}) = 0}$

$$\boxed{\mathrm{Tr}(\gamma_5 \not{a}\not{b}\not{c}\not{d}) = -4i\, \epsilon_{\alpha\beta\gamma\delta}\, a^\alpha b^\beta c^\gamma d^\delta}$$

$\epsilon^{\cdots} = +1$

for me

For our problem

$$\Gamma = \frac{1}{2p\cdot k}\, \not{\epsilon}'\not{k}\,\not{\epsilon} + \frac{1}{2p\cdot k'}\, \not{\epsilon}\not{k}'\not{\epsilon}'$$

We'll need some useful relations for the $\overline{\Gamma}$'s also--

note
$$\gamma^0 \not{a}^\dagger \gamma^0 = \gamma^0(\gamma^0 a_0 - \vec{\gamma}\cdot\vec{a})^\dagger\gamma^0 \qquad (a,\ \text{real})$$
$$= \gamma^0(\gamma^0 a_0 - \vec{\gamma}^\dagger\cdot\vec{a})\gamma^0$$

now $\gamma^i = \beta\alpha_i \quad \Rightarrow \quad \gamma^{i\dagger} = \alpha_i^\dagger\beta^\dagger$
$$= \alpha_i\beta$$
$$= -\beta\alpha_i = -\gamma^i$$

so
$$\gamma^0\not{a}^\dagger\gamma^0 = \gamma^0(\gamma^0 a_0 + \vec{\gamma}\cdot\vec{a})\gamma^0$$
$$= \gamma^0\gamma^0(\gamma^0 a_0 - \vec{\gamma}\cdot\vec{a})$$
$$= \gamma^0 a_0 - \vec{\gamma}\cdot\vec{a} = \not{a}$$

and so,
$$\gamma^0\not{a}^\dagger\gamma^0 \equiv \boxed{\overline{\not{a}} = \not{a}} \quad \text{or} \quad \boxed{\overline{\gamma^\mu} = \gamma^\mu}$$

note also
$$\boxed{\overline{i\gamma_5} = i\gamma_5}$$

so,
$$\overline{\not{\epsilon}'\not{k}\not{\epsilon}} = \gamma^0(\not{\epsilon}'\not{k}\not{\epsilon})^\dagger\gamma^0$$
$$= \gamma^0(\not{\epsilon}^\dagger\not{k}^\dagger\not{\epsilon}'^\dagger)\gamma^0$$
$$= \gamma^0\not{\epsilon}^\dagger\gamma^0\gamma^0\not{k}\gamma^0\gamma^0\not{\epsilon}'^\dagger\gamma^0$$
$$= \not{\epsilon}\not{k}\not{\epsilon}'$$

problem

generally
$$\boxed{\overline{\not{a}\not{b}\not{c}\cdots\not{z}} = \not{z}\cdots\not{c}\not{b}\not{a}}$$

so, also $\quad \not{\epsilon}\not{k}'\not{\epsilon}' = \not{\epsilon}'\not{k}'\not{\epsilon}$

and we find, using "the trick",

$$\sum_i\sum_f |T|^2 = \frac{e^4}{2}\,\mathrm{Tr}\left\{ (\not{p}'+m)\left[\frac{1}{2p\cdot k}\not{\epsilon}'\not{k}\not{\epsilon} + \frac{1}{2p\cdot k'}\not{\epsilon}\not{k}'\not{\epsilon}'\right](\not{p}+m)\right.$$

$$\left.\times\left[\frac{1}{2p\cdot k}\not{\epsilon}\not{k}\not{\epsilon}' + \frac{1}{2p\cdot k'}\not{\epsilon}'\not{k}'\not{\epsilon}\right]\right\}$$

$$= \frac{e^4}{2}\,\mathrm{Tr}\left\{ \frac{1}{(2p\cdot k)^2}(\not{p}'+m)\not{\epsilon}'\not{k}\not{\epsilon}(\not{p}+m)\not{\epsilon}\not{k}\not{\epsilon}'\right.$$

$$+ \frac{1}{(2p\cdot k')^2}(\not{p}'+m)\not{\epsilon}\not{k}'\not{\epsilon}'(\not{p}+m)\not{\epsilon}'\not{k}'\not{\epsilon}$$

$$+ \frac{1}{(2p\cdot k)(2p\cdot k')}(\not{p}+m)\not{\epsilon}'\not{k}\not{\epsilon}(\not{p}+m)\not{\epsilon}'\not{k}'\not{\epsilon}$$

$$\left.+ \frac{1}{(2p\cdot k)(2p\cdot k')}(\not{p}'+m)\not{\epsilon}\not{k}'\not{\epsilon}'(\not{p}+m)\not{\epsilon}\not{k}\not{\epsilon}'\right\}$$

$$= \frac{e^4}{2}\left\{\frac{\mathrm{Tr}(A)}{(2p\cdot k)^2} + \frac{\mathrm{Tr}(B)}{(2p\cdot k')^2} + \frac{\mathrm{Tr}(C)}{(2p\cdot k)(2p\cdot k')} + \frac{\mathrm{Tr}(D)}{(2p\cdot k)(2p\cdot k')}\right\}$$

$$\mathrm{Tr}(A) = \mathrm{Tr}\left[(\not{p}'+m)\not{\epsilon}'\not{k}\not{\epsilon}(\not{p}+m)\not{\epsilon}\not{k}\not{\epsilon}'\right]$$

$$= \mathrm{Tr}\left[\not{p}'\not{\epsilon}'\not{k}\not{\epsilon}(\not{p}+m)\not{\epsilon}\not{k}\not{\epsilon}' + m\not{\epsilon}'\not{k}\not{\epsilon}(\not{p}+m)\not{\epsilon}\not{k}\not{\epsilon}'\right]$$

$$= \mathrm{Tr}\left[\not{p}'\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}\not{k}\not{\epsilon}' + m\not{p}'\not{\epsilon}'\not{k}\not{\epsilon}\not{\epsilon}\not{k}\not{\epsilon}'\right.$$

$$\left.+ m\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}\not{k}\not{\epsilon}' + m^2\not{\epsilon}'\not{k}\not{\epsilon}\not{\epsilon}\not{k}\not{\epsilon}'\right]$$

also note $\quad \not{\epsilon}\not{\epsilon} = \epsilon^2 \quad$ the last term $\quad m^2\epsilon^2\not{\epsilon}'\not{k}\not{k}\not{\epsilon}'$

$$= m^2\epsilon^2k^2\not{\epsilon}'\not{\epsilon}'$$

$$= m^2\epsilon^2k^2\not{\epsilon}'^2$$

but, $k^2 = 0$ for our real photons.

first term $= Tr[\not p' \not\epsilon' \not k \not\epsilon \not p \not\epsilon \not k \not\epsilon']$

$= Tr[\not p' \not\epsilon' \not\epsilon \not k \not p \not k \not\epsilon \not\epsilon']$

since $\{\not k, \not\epsilon\} = 2k\cdot\epsilon = 0 \Rightarrow \not k \not\epsilon = -\not\epsilon \not k$

also, $\not k \not p \not k = \not k(-\not k \not p + 2p\cdot k)$

$= -\underbrace{\not k \not k \not p}_{0} + 2\not k\, p\cdot k$

$= 2\not k\, p\cdot k$

$Tr(A) = 2p\cdot k\, Tr(\not p' \not\epsilon' \not\epsilon \not k \not\epsilon \not\epsilon')$

$= -2p\cdot k\, Tr(\not p' \not\epsilon' \not k \not\epsilon \not\epsilon \not\epsilon')$

$\epsilon^2 = -\vec{\epsilon}\cdot\vec{\epsilon} = -1$

so $Tr(A) = 2p\cdot k\, Tr(\not p' \not\epsilon' \not k \not\epsilon')$ $\overset{-1}{''}$

$= 2p\cdot k\{4(p'\cdot\epsilon' k\cdot\epsilon' - p'\cdot k\, \epsilon'\cdot\epsilon' + p'\cdot\epsilon'\, \epsilon'\cdot k)$

$= 8p\cdot k\,(p'\cdot k + 2p'\cdot\epsilon' k\cdot\epsilon')$

remember that eventually the target electron is going to be at rest so, $p\cdot\epsilon' = p\cdot\epsilon = 0$ ... so do that now to save work (don't generally evaluate in a frame until later).

Then

$p'\cdot\epsilon' = \underbrace{(k+p-k')}_{\text{know I have the } \delta \text{ function}}\cdot\epsilon' = k\cdot\epsilon' + \underbrace{p\cdot\epsilon'}_{\overset{||}{0}\ \text{rest}} - \underbrace{k'\cdot\epsilon'}_{\overset{||}{0}\ \text{transverse}}$

$Tr(A) = 8p\cdot k\,(p'\cdot k + 2(k\cdot\epsilon')^2)$

Similarly,
$$Tr(B) = 8p\cdot k'\left[p'\cdot k' - 2(k'\cdot\varepsilon)^2\right]$$
$$Tr(C) = 8p\cdot k\, p\cdot k'\left[2(\varepsilon'\cdot\varepsilon)^2 - 1\right] - 8(k\cdot\varepsilon')^2 p\cdot k' + 8(k'\cdot\varepsilon)^2 k\cdot p$$
$$Tr(D) = Tr(C)$$

and we get,

$$\sum_i \sum_f |T|^2 = \frac{e^4}{2}\left\{ \frac{8p\cdot k}{(2p\cdot k)^2}\left[p'\cdot k + 2(k\cdot\varepsilon')^2\right] + \frac{8p\cdot k'}{(2p\cdot k')^2}\left[p'\cdot k' - 2(k'\cdot\varepsilon)^2\right] + \frac{2}{2p\cdot k\, 2p\cdot k'}\left\{8p\cdot k\, p\cdot k'\left[2(\varepsilon'\cdot\varepsilon)^2 - 1\right] - 8(k\cdot\varepsilon')^2 p\cdot k' + 8(k'\cdot\varepsilon)^2 k\cdot p\right\}\right\}$$

$$= 4e^4\left\{ \frac{1}{4p\cdot k}\left[p'\cdot k + 2(k\cdot\varepsilon')^2\right] + \frac{1}{4p\cdot k'}\left[p'\cdot k' - 2(k'\cdot\varepsilon)^2\right] + \frac{1}{2}\left[2(\varepsilon'\cdot\varepsilon)^2 - 1\right] - \frac{1}{2}\frac{(k\cdot\varepsilon)^2}{p\cdot k} + \frac{1}{2}\frac{(k'\cdot\varepsilon)^2}{p\cdot k'}\right\}$$

$$= e^4\left\{ \frac{p'\cdot k}{p\cdot k} + \cancel{\frac{2(k\cdot\varepsilon')^2}{p\cdot k}} + \frac{p'\cdot k'}{p\cdot k'} - \cancel{\frac{2(k'\cdot\varepsilon)^2}{p\cdot k'}} + 2\left[2(\varepsilon'\cdot\varepsilon)^2 - 1\right] - \cancel{\frac{2(k\cdot\varepsilon')^2}{p\cdot k}} + \cancel{\frac{2(k'\cdot\varepsilon)^2}{p\cdot k'}}\right\}$$

Again, using overall 4momentum conservation (standard trick)..

$$p+k = p'+k'$$
$$(p-k')^2 = (p'-k)^2$$
$$\cancel{p^2} + \cancel{k'^2}^{\,0} - 2p\cdot k' = \cancel{p'^2} + \cancel{k^2}^{\,0} - 2p'\cdot k$$
$$p\cdot k' = p'\cdot k$$

also, $(p+k)^2 = (p'+k')^2 \longrightarrow p\cdot k = p'\cdot k'$

so,
$$= e^4\left\{ \frac{p'\cdot k}{p\cdot k} + \frac{p'\cdot k'}{p\cdot k'} + 2\left[2(\varepsilon'\cdot\varepsilon)^2 - 1\right]\right\}$$
$$= e^4\left\{ \frac{p\cdot k'}{(p\cdot k)} + \frac{p\cdot k}{(p\cdot k')} + 2\left[2(\varepsilon'\cdot\varepsilon)^2 - 1\right]\right\}$$

now enforce the frame,

$$p\cdot k' = m\omega'$$
$$p\cdot k = m\omega$$

$$= e^4\left\{ \frac{m\omega'}{m\omega} + \frac{m\omega}{m\omega'} + 2\left[2(\varepsilon'\cdot\varepsilon)^2 - 1\right]\right\}$$
$$= e^4\left[\frac{(\omega-\omega')^2}{\omega\omega'} + 4(\varepsilon'\cdot\varepsilon)^2\right]$$