The absolute energy scale is <u>arbitrary</u>-- , so we can define the state of lowest energy to have value zero

$$\text{so} \qquad H|0\rangle = 0$$

This is the same as subtracting $\frac{1}{2}\hbar\omega$ from the radiation Hamiltonian, so:

$$\hat{H} = \sum_k \sum_\lambda \hbar\omega\, \hat{a}^+_{k\lambda}\, \hat{a}_{k\lambda} \quad .$$

$\longrightarrow$ ~~96a~~ 96b
438.1 & 438.2

Also, $\vec{A}$ is changed.. now

$$\hat{\vec{A}}(\vec{x},t) = \sum_k \sum_\lambda \sqrt{\frac{\hbar}{2\omega V}}\, c\left[\vec{\epsilon}_{k\lambda}\, \hat{a}_{k\lambda}\, e^{i\vec{k}\cdot\vec{x}} + \vec{\epsilon}^{\,*}_{k\lambda}\, \hat{a}^+_{k\lambda}\, e^{-i\vec{k}\cdot\vec{x}}\right]$$

is the box-normalized photon field <u>operator</u>.

What do these operators operate on? A many-field space called Occupation Number Representation or Fock Space.... clearly motivated by the harmonic oscillator model of Dirac.

In Fock Space, the state vector

$$|\psi\rangle = |\, n(k_1), n(k_2), n(k_3) \cdots n(k_i) \cdots \rangle$$

each "slot" denotes the number of quanta which have a given momentum.

The state vector is an eigenstate of combinations of the a's —

$\hat{a}(h_i)$ will annihilate (<u>absorb</u> in Dirac's original view) a quantum of momentum h_i and not affect any other states

remember the problem was one of a radiation field losing and gaining individual photons + Einstein's A's & B's ..

These operators do that

so
$$\hat{a}(h_i)|\psi\rangle = \hat{a}(h_i)|n(h_1), n(h_2), \ldots, n(h_i), \ldots\rangle$$
$$= \sqrt{n(h_i)}\,|n(h_1), n(h_2) \ldots, n(h_i)-1, \ldots\rangle$$

$\hat{a}^{+}(h_i)$ will create ("emit") a photon of momentum h_i

$$\hat{a}^{+}(h_i)|\psi\rangle = \hat{a}^{+}(h_i)|n(h_1), n(h_2), \ldots, n(h_i), \ldots\rangle$$
$$= \sqrt{n(h_i)+1}\,|n(h_1), n(h_2) \ldots, n(h_i)+1, \ldots\rangle$$

also:
$$\langle n(h_i)|\hat{a}^{+}(h_i) = \sqrt{n(h_i)}\,\langle n(h_i)-1|$$
$$\langle n(h_i)|\hat{a}(h_i) = \sqrt{n(h_i)+1}\,\langle n(h_i)+1|$$

clearly $|\psi\rangle$ is not an eigenstate of either a or a^{+}. However guided by the h.o.

$$\hat{a}(k_i)\,\hat{a}^{+}(k_i)\,|n(k_i)\rangle = \sqrt{n(k_i)+1}\;\hat{a}(k_i)\,|n(k_i)+1\rangle$$
$$= \sqrt{(n(k_i)+1)}\sqrt{n(k_i)+1}\;|n(k_i)\rangle$$

and $$\hat{a}^{+}(k_i)\hat{a}(k_i)\,|n(k_i)\rangle = \sqrt{n(k_i)}\;\hat{a}^{+}(k_i)\,|n(k_i)-1\rangle$$
$$= \sqrt{n(k_i)}\sqrt{n(k_i)-1+1}\;|n(k_i)\rangle$$
$$= n(k_i)\,|n(k_i)\rangle$$

so, $\hat{a}^{+}_{\lambda}(k_i)\,\hat{a}_{\lambda}(k_i) \equiv N_{\lambda}(k_i)$ -- the number operator, which counts the number of photons in momentum state k_i and with λ

note

so, $$\hat{H} = \sum_{k}\sum_{\lambda} \hbar\omega\, N_{\lambda}(k) \qquad \omega = |\vec{k}|c$$

and total energy is the sum of individual energies

note $\hat{H}\,|n_{\lambda}(k_i)\rangle = |n_{\lambda}(k_i)\rangle\,\hbar\omega_i \qquad \omega_i = |\vec{k}_i|c$,

Just as w/ ho, there is a ground state — the vacuum — in which there are no quanta.

$|\Psi_0\rangle = |0,0,\ldots 0,\ldots\rangle \equiv |0\rangle$ and unique normalization $\langle 0|0\rangle = 1$

so, $\hat{a}(k_i)\,|0\rangle = 0 \qquad N(k)\,|0\rangle = 0$

It is the state of lowest energy.

$$H|0\rangle = \sum\sum \hbar\omega\,\hat{a}^{+}\hat{a}\,|0\rangle = 0 \quad \text{or}$$

$$\langle 0|H|0\rangle \propto \langle 0|\hat{a}^{+}\hat{a}\,|0\rangle = 0$$

either way

Lecture 6 2/7/03

Suppose we know the energy of a state,

$$H|\psi\rangle = E|\psi\rangle$$

and make a new one $\hat{a}^+(k')|\dots n(k')\dots\rangle = \sqrt{n(k')+1}\,|\dots, n(k')+1, \dots\rangle$

what's the energy of it?

ie what's $H\,\hat{a}^+(k')|n\rangle \equiv H|n'\rangle$

Find. $$[\hat{H}, \hat{a}^+_{\lambda'}(k')] = \sum_k \sum_\lambda \hbar\omega\,[\hat{N}_\lambda(k), \hat{a}^+_{\lambda'}(k')]$$

add zero

$$= \sum\sum \hbar\omega\,[\,a^+ a a^{+\prime} - a^{+\prime} a^+ a \;\underbrace{- a^+ a^{+\prime} a + a^+ a^{+\prime} a}_{0}\,]$$

$$= \sum\sum \hbar\omega \left\{ \hat{a}^+_\lambda(k)\underbrace{[\hat{a}_\lambda(k), \hat{a}^+_{\lambda'}(k')]}_{=\delta_{kk'}\delta_{\lambda\lambda'}} + \underbrace{[\hat{a}^+_\lambda(k), \hat{a}^+_{\lambda'}(k')]}_{=0}\hat{a}_\lambda(k) \right\}$$

$$[\hat{H}, \hat{a}^+_{\lambda'}(k')] = \hat{H}\hat{a}^+_{\lambda'}(k') - \hat{a}^+_{\lambda'}(k')\hat{H} = \hbar\omega'_{k'}\,\hat{a}^+_{\lambda'}(k')$$

so $$\hat{H}\hat{a}^+_{\lambda'}(k') = \hat{a}^+_{\lambda'}(k')H + \hbar\omega'_{k'}\,\hat{a}^+_{\lambda'}(k')$$

and $$\hat{H}\hat{a}^+_{\lambda'}(k')|\dots n_{\lambda'}(k')\dots\rangle = \hat{a}^+_{\lambda'}(k')H|\dots n_{\lambda'}(k')\dots\rangle + \hbar\omega'_{k'}\,\hat{a}^+_{\lambda'}(k')|\dots n_{\lambda'}(k')\dots\rangle$$

$$= \hat{a}^+_{\lambda'}(k')|\dots n_{\lambda'}(k')\dots\rangle E + \hat{a}^+_{\lambda'}(k')|\dots n_{\lambda'}(k')\dots\rangle\hbar\omega'_{k'}$$

$$= (E + \hbar\omega'_{k'})|n'\rangle$$

– the energy of the new state increases by $\hbar\omega'$.

So, now to radiation. A standard treatment was to take the Hamiltonian (non-relativistic) and through the minimal subtraction method $\vec{p} \to \vec{p} - e\vec{A}/c$

now the radiation field operator

A many body Hamiltonian

$$H = \sum_{i=\text{electrons}} \frac{\vec{p}_i^{\,2}}{2m_i} + \frac{1}{2} \sum_{i \neq j} \underbrace{\frac{e_i e_j}{|\vec{x}_i - \vec{x}_j|}}_{V}$$

add $\vec{A}$

$$= \sum_i \frac{(\vec{p}_i - \frac{e_i}{c}\vec{A})(\vec{p}_i - \frac{e_i}{c}\vec{A})}{2m_i} + V + \int \frac{\hat{E}\cdot\hat{E}^* + \hat{B}\cdot\hat{B}^*}{8\pi} d^3x$$

the interaction $\underline{H_{int}}$ inside

$$\hat{H}_{int} = \sum_i \left\{ \frac{-e_i}{2m_i c} \left[\hat{\vec{p}}_i \cdot \hat{\vec{A}}(\vec{x}_i, t) + \hat{\vec{A}}(\vec{x}_i, t) \cdot \hat{\vec{p}}_i \right] + \frac{e_i^2}{2m_i c^2} \hat{\vec{A}}(x_i, t) \cdot \hat{\vec{A}}(\vec{x}_i, t) \right\}$$

will not contribute

$\vec{A}$ Fock operator which acts on <u>many</u> <u>photon state</u> at $\vec{x}_i$

$\vec{p}$ is a differential operator which operates on <u>everything</u> to the right.

so $\quad \vec{p}_i \cdot \vec{A}\, |\psi\rangle = (\vec{p}_i \cdot \vec{A})|\psi\rangle + \vec{A}\cdot\vec{p}_i|\psi\rangle$

in the Coulomb gauge $\quad \vec{p}\cdot\vec{A} \sim \vec{\nabla}\cdot\vec{A} = 0$

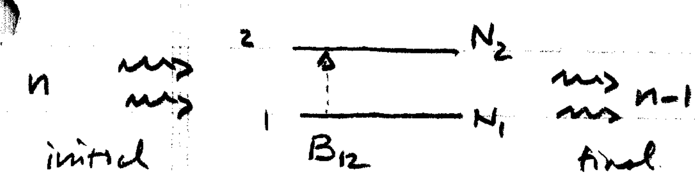
so the term [ ]

can be written $\left[\; 2\vec{A}(\vec{x}_i, t)\cdot\vec{p}_i \;\right]$

$$|\psi\rangle = |\text{many atomic electrons}\rangle \otimes |\text{many photons}\rangle$$

for one two level system exposed to a bath of radiation (leading to stimulated absorption or emission)

$$|\psi_{\text{initial}}\rangle = |A_{\text{initial}}\rangle \otimes |\cdots n_{k\lambda}\cdots\rangle$$



frequency distribution characterized by $\rho(\nu)$

The transition probability is obviously proportional to

$$|\langle\psi_{\text{final}}|\,H_{\text{int}}\,|\psi_{\text{initial}}\rangle|^2$$

treated as a 1st order perturbation.

suppose that only photons with $\omega_k = |\vec{k}|c$ & that only one atom makes the transition

The process is

$$\langle\ldots, n_{k\lambda}-1, \ldots|\langle A_{\text{final}}|\,H_{\text{int}}\,|A_{\text{initial}}\rangle|\cdots n_{k\lambda}\cdots\rangle$$

so we need the parts of $\vec{A}\cdot\vec{p}$ and $\vec{A}\cdot\vec{A}$ which annihilate 1 photon

$$A \sim a + a^{+}$$

only need this piece why? $\Rightarrow$ $a^{+}|\ldots n\rangle = |n+1\rangle$, $\langle n-1|n+1\rangle = 0$

note that $\vec{A}\vec{A}$ will create and annihilate 0 or ±2 photons X

$$\langle \ldots, n_{k\lambda}-1, \ldots ; A_{final} \,|\, \frac{-e}{2mc}\left(\sqrt{\frac{\hbar}{2\omega V}}\; \vec{\epsilon}_{k\lambda}\, a_{k\lambda}(0)\, e^{-i\omega_k t}\, e^{i\vec{k}\cdot\vec{x}} \cdot \vec{p}\right) | A_{initial} ; \ldots n_{k\lambda} \ldots \rangle$$

so $$a_{k\lambda}(0)\,|\ldots n_{k\lambda}\rangle = \sqrt{n_{k\lambda}}\;|\ldots n_{k\lambda}-1\ldots\rangle$$

and there are no other fock space operators, so

$$\langle \ldots n-1 \ldots | \ldots n-1 \ldots \rangle = 1$$

$$\frac{-e}{2mc}\sqrt{\frac{\hbar\, n_{k\lambda}}{2\omega V}}\;\langle A_{final} |\, e^{i\vec{k}\cdot\vec{x}}\; \vec{\epsilon}_{k\lambda}\cdot\vec{p}\; | A_{initial}\rangle\, e^{-i\omega_k t}$$

Now, what would have done for a <u>classical</u> EM field?

you would have used an harmonic perturbation like

$$\vec{A}(\vec{x},t) = \int \underline{A(\omega)}\, \vec{\epsilon}\; e^{-i\omega t + \vec{x}\cdot\vec{k}}\, d\omega$$

(Merzbacher)

why didn't you see this before? Because in the real field-theoretic approach $A(\omega)$ contains the $\sqrt{n_{k\lambda}}$ and you never see it.. you get exactly the same answer.. for arbitrarily <u>small</u> or <u>large</u> numbers of photons, either way.

Emission is a whole other story

$$\langle \ldots, n_{k\lambda}+1, \ldots; A_{final} \mid H_{int} \mid A_{initial}; \ldots n_k \ldots \rangle$$

$A \sim a + a^\dagger$

↖ want this one

$$\frac{-e}{2mc}\sqrt{\frac{\hbar}{2\omega V}}\,\sqrt{n_{k\lambda}+1}\,\langle A_{final} \mid e^{-i\vec{k}\cdot\vec{x}}\,\vec{\epsilon}^{\,*}_{k\lambda}\cdot\vec{p} \mid A_{initial}\rangle e^{i\omega_k t}$$

ahah! see this?

For large numbers of photons, $\sqrt{n+1} \sim \sqrt{n}$ and you have the regular, classical result. <u>But</u>, for very small numbers of photons, you get a decidedly non-classical result — <u>that's</u> stimulated emission. → also influence the flux.

In a field-theoretic, which means many body, formalism you get both stimulated and spontaneous emission.

Reminder for classical mechanics.

$$\dot{p} = -\frac{\partial H}{\partial q} \qquad \dot{q} = \frac{\partial H}{\partial p}$$ and Poisson Brackets:

$$\{A, B\}_{PB} \equiv \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$$

Using Ham. Eq., n

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial p}\dot{p} + \frac{\partial A}{\partial q}\dot{q}$$

$$= \frac{\partial A}{\partial t} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial A}{\partial q}\frac{\partial H}{\partial p}$$

$$= \frac{\partial A}{\partial t} + \{A, H\}_{PB}$$

in particular $$\{q, p\} = -\cancel{\frac{\partial p}{\partial q}\frac{\partial q}{\partial p}}^{\,0} + \frac{\partial p}{\partial p}\frac{\partial q}{\partial q}$$

$$= 1$$

all referenced in canonical quantization to reg. QM.

Now, for Functionals, $A(\phi,\pi)$, $B(\phi,\pi)$ define

$$\{A,B\}_{PB} = \int d^3x \left(\frac{\delta A}{\delta\phi(x)} \frac{\delta B}{\delta\pi(x)} - \frac{\delta A}{\delta\pi(x)} \frac{\delta B}{\delta\phi(x)} \right)$$

In the same spirit

$$\dot{A}(t) = \int d^3x \left(\frac{\delta A(t)}{\delta\phi(x)} \underbrace{\dot{\phi}(x)}_{\frac{\delta H}{\delta \pi}} + \frac{\delta A(t)}{\delta\pi(x)} \underbrace{\dot{\pi}(x)}_{-\frac{\delta H}{\delta \phi}} \right)$$

Hamilton's Equations.

$$= \int d^3x \left(\frac{\delta A(t)}{\delta\phi(x)} \frac{\delta H}{\delta\pi} - \frac{\delta A(t)}{\delta\pi(x)} \frac{\delta H}{\delta\phi} \right)$$

$$\dot{A}(t) = \{A,H\}_{PB}$$

An important special case... at the same times –

$$\phi(\vec{x}\,t) = \int d^3y\, \phi(\vec{y}\,t)\, \delta^3(\vec{x}-\vec{y})$$

$$\frac{\delta\phi(x,t)}{\delta\phi(y,t)} = \delta^3(\vec{x}-\vec{y}) \quad \text{since} \quad \delta\phi(xt) \equiv \int d^3x\, \frac{\delta\phi(x,t)}{\delta\phi(yt)}\delta(yt)$$

likewise $\quad \dfrac{\delta\pi(x,t)}{\delta\pi(y,t)} = \delta^3(x-y) \qquad \dfrac{\delta\pi(x\,t)}{\delta\phi(y\,t)} = \dfrac{\delta\phi}{\delta\pi} = 0$

so. $$\dot\phi(x t) = \{ \phi(x t), H(t) \}_{PB}$$

$$= \int d^3y \; \frac{\delta\phi(xt)}{\delta\phi(yt)} \frac{\delta H}{\delta\pi(y,t)} - \frac{\delta\phi(xt)}{\delta\pi(yt)} \frac{\delta H}{\delta\phi(yt)}$$

0

$$= \int d^3y \; \underbrace{\frac{\delta\phi(xt)}{\delta\phi(yt)}}_{\delta(x-y)} \frac{\delta H(t)}{\delta\pi(yt)}$$

$$\dot\phi(x,t) = \frac{\delta H(t)}{\delta\pi(xt)} \qquad \checkmark$$

and

$$\dot\pi(x,t) = -\frac{\delta H(t)}{\delta\phi(xt)} \qquad \checkmark$$

Also,

$$\{\phi(xt), \phi(y,t)\}_{PB} = \int d^3z \; \underbrace{\frac{\delta\phi(xt)}{\delta\phi(zt)}}_{\delta(x-z)} \underbrace{\frac{\delta\pi(yt)}{\delta\pi(zt)}}_{\delta(y-z)}$$

$$= \delta(x-y) \quad @ \text{ equal times.}$$

The idea is now in parallel —

<u>Classical Mechanics</u> $\longrightarrow$ <u>quantum mechanics</u>

q & p $\rightarrow$ satisfy Ham Eq $\longrightarrow$ operators

$\{q,p\} = 1$ $\longrightarrow$ commutators $[p,q] = i$

<u>classical field theory</u> $\longrightarrow$ <u>quantum field theory</u>

ϕ and π $\rightarrow$ satisfy Ham Eq $\longrightarrow$ operators

$\{\phi, \pi\}_{PB} = \delta(\quad)$ $\longrightarrow$ commutators $[\phi, \pi] = i\delta(\quad)$

Motivation? The Harmonic Oscillator —

<u>Now</u>, it's a many-body theory with <u>states</u> corresponding to the normal modes of an harmonic distribution.

We see the operator-character of the ϕ by

$$a, a^{\dagger} \longrightarrow \phi = \underbrace{a + a^{\dagger}}_{\text{operators}} \longrightarrow \;\Rightarrow\; \phi\text{-operators}$$