Lecture 18 Intro-Wick

Let me summarize and look ahead, as things get a bit messy for a bit. This is a $\boxed{\text{schematic introduction}}$ to the next steps...

We've established that the S matrix contains the physics and is an iterative combination of operators, built into the interaction Hamiltonian or Lagrangian.

$$S^{(0)} = 1$$
$$S^{(1)} = -i\int_{-\infty}^{\infty} dt_1 \mathcal{H}_I(t_1)$$
$$S^{(2)} = (-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 P(\tilde{\mathcal{H}}_I(t_1)\tilde{\mathcal{H}}_I(t_2))$$
$$\vdots$$

forget time-ordering for this overview!

What we need are the matrix elements, usually to a given order in some parameter within $\mathcal{H}_I$, between some initial state $|m\rangle$ and some final state $\langle n|$

$$\langle n| S^{(0)}|m\rangle = \langle n|m\rangle = 0 \qquad \text{no scattering.}$$

$$\langle n| S^{(1)}|m\rangle = -i\int dt_1 \langle n|\tilde{\mathcal{H}}_I(t_1)|m\rangle$$

express in terms of Schrödinger picture

$$= -i\int dt_1 \langle n| e^{iH_0t}\mathcal{H}_I e^{-iH_0t}|m\rangle$$
$$= -i\langle n|\mathcal{H}_I|m\rangle \int_{-\infty}^{\infty} dt\, e^{i(E_n-E_m)t}$$

$$= -2\pi i \langle n|\mathcal{H}_I|m\rangle \delta(E_n-E_m)$$

18 209

which is the Golden Rule

continue (no time ordering ..)

$$\langle n|S^2|m\rangle = (-i)^2 \langle n| \int dt_1 \int dt_2\, \tilde{\mathcal{H}}_I(t_1)\tilde{\mathcal{H}}_I(t_2)|m\rangle$$

$$= (-i)^2 \sum_\ell \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \langle n|\tilde{\mathcal{H}}_I(t_1)|\ell\rangle\langle \ell|\tilde{\mathcal{H}}_I(t_2)|m\rangle$$

$$= (-i)^2 \sum_\ell \langle n|\mathcal{H}_I|\ell\rangle\langle \ell|\mathcal{H}_I|m\rangle$$

$$\times \int_{-\infty}^{\infty} dt_1\, e^{i(E_n-E_\ell)t_1} \underbrace{\int_{-\infty}^{t_1} dt_2\, e^{i(E_\ell-E_m)t_2}}$$

oscillates .. add $i\eta$ for convergence

$$\frac{-i e^{i(E_\ell-E_m-i\eta)t_1}}{E_\ell-E_m-i\eta} = \frac{i e^{i(E_\ell-E_m)t_1}}{E_m-E_\ell}$$

$$\langle n|S^2|m\rangle = (-i)^2 \sum_\ell \langle n|\mathcal{H}_I|\ell\rangle\langle \ell|\mathcal{H}_I|m\rangle$$

$$\frac{1}{E_m-E_\ell} \int dt_1\, e^{i(E_n-E_\ell)t_1}\, i e^{i(E_\ell-E_m)t_1}$$

$$= (-i)^2 \sum_\ell \langle n|\mathcal{H}|\ell\rangle\langle \ell|\mathcal{H}_I|m\rangle$$

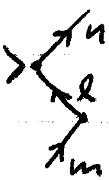
$$\times \frac{i}{E_m-E_\ell+i\eta} \int_{-\infty}^{\infty} dt_1\, e^{i(E_n-E_m)t_1}$$

no ℓ dependence

$$= -2\pi i \sum_\ell \frac{\langle n|\mathcal{H}_I|\ell\rangle\langle \ell|\mathcal{H}_I|m\rangle\, \delta(E_n-E_m)}{E_m-E_\ell}$$

These are the things represented by Feynman diagrams.

$S^{(1)} \Rightarrow$ 

$S^{(2)} \Rightarrow$ $\langle l|\mathcal{H}_I|n\rangle$  $\langle m|\mathcal{H}_I|l\rangle$

$\sum$ intermediate states.— no energy conservation.— only $\delta(E_n - E_m)$ for initial & final

go back.— now forget exponentials. . schematically, just operators

$$S = \langle n|m\rangle - i2\pi\,\delta(E_n - E_m)\Big[\langle n|\mathcal{H}_I|m\rangle + \sum_l \frac{\langle n|\mathcal{H}_I|l\rangle\langle l|\mathcal{H}_I|m\rangle}{E_m - E_l} + \dots\Big]$$

$$= \delta_{nm} - i2\pi\,\delta(E_n - E_m)\Big[V_{nm} + \sum_l \frac{V_{nl}V_{lm}}{E_m - E_l} + \dots\Big]$$

now E_l is the eigenvalue of $\underset{\sim}{H_0}|l\rangle = E_l|l\rangle$

notice $\quad (E_m - H_0)|l\rangle = (E_m - E_l)|l\rangle$

operate with

$$(E_m - H_0)^{-1} \rightarrow \quad |l\rangle = (E_m - E_l)(E_m - H_0)^{-1}|l\rangle$$

or $\times (E_m - E_l)^{-1}$

$$(E_m - E_l)^{-1}|l\rangle = (E_m - H_0)^{-1}|l\rangle$$

So, write, again somewhat schematically

$$T = 2\pi\delta(E_n - E_m)\langle n|\,(i\mathcal{H}_I) + (i\mathcal{H}_I)\,\frac{i}{E_m - H_0}\,(i\mathcal{H}_I) \cdots |m\rangle$$

what's this?

Suppose we write the Schrödinger equation like

$$(H_0 + H_I)|\psi\rangle = E|\psi\rangle$$

$$i\,(H_0 - E)|\psi\rangle = \underbrace{H_I|\psi\rangle}_{\text{interaction}}$$

$$\frac{i}{E_m - H_0} \text{ is } \frac{1}{i(\;\circ\;)}$$ the propagator

With this recipe, we can just read them off in momentum space:

K.G. $$(\Box + m^2)\phi = V\phi$$

K.G. prop: $$\frac{1}{i(-p^2 + m^2)} = \frac{i}{p^2 - m^2}$$

Dirac. $$(\not{p} - m)\psi = V\psi$$

Dirac prop: $$\frac{1}{i(\not{p} - m)} = \frac{i}{\not{p} - m} = \frac{i(\not{p} + m)}{p^2 - m^2}$$

Remember $\not{p}+m$ is the form of the completeness relation $\Sigma u\bar{u}$, and <u>Any</u> propagator for any spin can be written

$$\frac{i \Sigma \psi \bar{\psi}}{p^2 - m^2}$$

Spin 1: $g^{\mu\nu} \Box A_\nu = j^\mu$

so propagator for photon: $-\frac{i g^{\mu\nu}}{q^2}$

massive: $[g^{\mu\nu}(\Box + M^2) - \partial^\mu \partial^\nu] B_\nu = 0$

$$\frac{i\left(-g^{\mu\nu} + \frac{p^\mu p^\nu}{M^2}\right)}{p^2 - M^2}$$

What's this about? This is a criminally oversimplified view of "propagator theory" (see Volume I of Bjorken and Drell). What we're really doing is solving a wave equation for which Green's Function techniques are necessary.

For example, for the Dirac Equation

$$\underbrace{(i\not\partial - m)\psi}_{\substack{\text{free}\\\text{particle}\\\text{stuff}}} = \underbrace{-e\not A\psi}_{\text{interaction or source stuff}}$$

one solves the point source problem

$$(i\not\partial - m)\,G_F = \delta^4(x-x')$$

and then

$$\psi(x) = -e\int d^4x'\, G_F(x-x')\,\not A(x')\,\psi(x')$$

Fourier transform $G_F(x-x')$ into momentum space

$$G_F(x-x') = \frac{1}{(2\pi)^4}\int S_F(p)\, e^{-ip\cdot(x-x')}\, d^4p$$

Substituting, -- going to momentum space

$$\frac{1}{(2\pi)^4}\int (\not p - m)\, S_F(p)\, e^{-ip\cdot(x-x')}\, d^4p = \frac{1}{(2\pi)^4}\int e^{ip\cdot(x-x')}\, d^4p$$

or..

$$(\not p - m)\, S_F(p) = 1$$

$$S_F(p) = \frac{1}{\not p - m} = \frac{\not p + m}{p^2 + m^2}$$

So, that term in $S^{(2)}$ is just the Fourier Transform of the Green's Function associated with the Dirac Equation.

We'll see this develop more formally, but I wanted the high spots to be noted before the formalism descends!