Let's consider some examples.

Imagine an infinitesimal space time shift — translation of the coordinate system:

$$x^{\mu\prime} = x^\mu + \xi^\mu \qquad \xi^\mu \to 0$$

$$\delta x^\mu = \xi^\mu \text{ , a constant}$$

This is inherently a part of all physics → Einstein's 2nd Postulate of Relativity → homogeneity of all spacetime and so $\delta S = 0$ ~~is not~~ is expected.

The shape of fields shouldn't change under such a transformation, so

$$\varphi'(x') = \varphi(x) \qquad \text{for just a translation.} \qquad \delta\varphi = 0$$

(This is not the infinitesimal Lorentz transformation, right?)

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad \Rightarrow \quad \phi'(x') = \phi(\Lambda^{-1}x')$$

$$\to x^\mu + \delta\omega^{\mu\nu} x_\nu$$

The Poincaré group encompasses both L.T. and shift.

$$\cancel{J_\mu(x) = \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)} \frac{\partial \varphi}{\partial x^\nu} - g_{\mu\nu} \mathcal{L} \right) \xi^\mu}$$

$$J^\mu(x) = \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)} \frac{\partial \varphi}{\partial x_\nu} - g^{\mu\nu} \mathcal{L} \right) \xi_\nu$$

The traditional way to represent this is

$$\delta S = 0 \Rightarrow \xi_\nu \frac{\partial}{\partial x^\mu} \Theta^{\mu\nu} = 0$$

where $\Theta^{\mu\nu} = \dfrac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)} \dfrac{\partial \varphi}{\partial x_\nu} - g^{\mu\nu} \mathcal{L}(x)$

which is the "Noether current" ~~also~~.

Look at 3 space integral

$$\int \Theta^{\mu\nu} d^3x = \int \frac{\partial \Theta^{0\nu}}{\partial t} d^3x - \underbrace{\int \frac{\Theta^{i\nu}}{\partial x^i} d^3x}_{\vec{\nabla} \cdot \vec{\Theta}\, d^3x} = 0$$

so, $\Theta^{i\nu} dA_i = 0$ @ surface

$$\frac{\partial}{\partial x^\mu} \int \Theta^{\mu\nu} d^3x \rightarrow \frac{\partial}{\partial t} \int \Theta^{0\nu} d^3x = 0$$ which is the constant of the motion

N.B. $\Theta^{0\nu}$ does not transform like a 4-vector, $\int \Theta^{0\nu} d^3x$ does.

What is it?

$$\Theta^{00} = \frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^0)} \frac{\partial \varphi}{\partial x_0} - g^{00}\mathcal{L}$$

$$= \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \dot{\varphi} - \mathcal{L}$$

$$\Theta^{00} = \pi\dot{\varphi} - \mathcal{L}$$

when recalling $H = \dot{q}\frac{\partial L}{\partial \dot{q}} - L$ we can see that

$\Theta^{00} = \mathcal{H}$, the Hamiltonian Density

$\Theta^{0\nu}$ is the conserved "Noether ~~Charge~~ Current" of which the $\nu=0$ component is the conserved charge -- H.

So, the Noether conservation law is Energy.

Then, $\int \Theta^{0\nu} d^3x = P^\nu$ the 4-momentum, the Generator of Space Time translations

Also, P^ν is one of the generators of the Poincare Group.

— $\Theta^{\mu\nu}$ as a whole is the Stress Energy Tensor.

→ under a L.T., angular momentum is a part of what's conserved, and so the whole of $\Theta^{\mu\nu}$ is ~~[illegible]~~ involved.

The other extreme is a symmetry of the fields themselves -- phase rotations

From $\mathcal{L} = \frac{1}{2}\left(\partial_\mu \varphi \, \partial^\mu \varphi - m^2 \varphi^2 \right)$

we can see immediately that a transformation like

$$\varphi(x) \longrightarrow \varphi'(x) = e^{i\alpha} \varphi(x)$$

does not leave
~~leaves~~ $\mathcal{L}$ invariant. What can Noether's Theorem say about this? Nothing -- it's not a physical thing of importance.

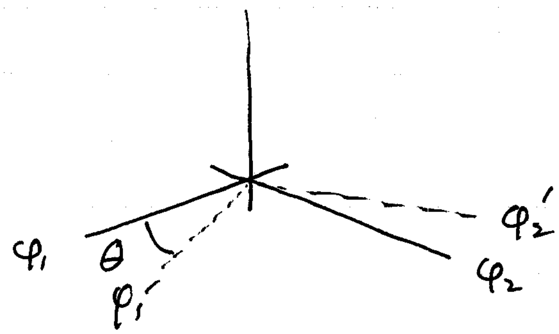
But, what about a multi component theory - where $\varphi = \varphi_i$ a set of fields. Try 2 dimensional fields:

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{2} \left[\frac{\partial \varphi_i}{\partial x^\mu} \frac{\partial \varphi_i}{\partial x_\mu} - m^2 \varphi_i^2 \right]$$

degenerate in mass, $\vec{\varphi} = \{ \varphi_1, \varphi_2 \}$
which would give separate K-G. equations for each component

$$\Box \varphi_j + m^2 \varphi_j = 0 \qquad j = 1 \text{ or } 2.$$

Now there <u>is</u> an obvious rotational symmetry.



$$\left.\begin{aligned}\varphi_1 \rightarrow \varphi_1' &= \varphi_1 \cos\theta - \varphi_2 \sin\theta \\ \varphi_2 \rightarrow \varphi_2' &= \varphi_1 \sin\theta + \varphi_2 \cos\theta\end{aligned}\right\} \quad \varphi_1 \text{ and } \varphi_2 \text{ \underline{mix}}$$

An easy way to put this into a framework of interest to us...

$$\left.\begin{aligned}\varphi &= \sqrt{1/2}\,(\varphi_1 + i\varphi_2) \\ \varphi^* &= \sqrt{1/2}\,(\varphi_1 - i\varphi_2)\end{aligned}\right\}$$

φ and φ^* would have the same K-G equation as φ_1 and φ_2.

$$\mathcal{L} = \left(\frac{\partial \varphi}{\partial x^\mu}\right)^*\left(\frac{\partial \varphi}{\partial x_\mu}\right) - m^2 \varphi^* \varphi$$

does this.

Now, the above rotation can be thought of as a phase rotation.

$$\varphi \rightarrow \varphi' = e^{-i\alpha}\varphi$$
$$\varphi^* \rightarrow \varphi^{*\prime} = e^{i\alpha}\varphi^*$$

To be related to a Lie Group, we need to be able to build from infinitesimal transformations

$$\varphi \to \varphi' = (1 - i\alpha)\varphi$$
$$\varphi^* \to \varphi'^* = (1 + i\alpha)\varphi.$$

So, $\varphi' = \varphi + \delta\varphi$ where $\varphi = -i\alpha\varphi.$

$\delta x^\mu = 0$

$\mathcal{H}$ can be gotten from $\quad \mathcal{L} = \frac{\partial\varphi^*}{\partial t}\frac{\partial\varphi}{\partial t} - \vec\nabla\varphi^*\cdot\vec\nabla\varphi - m^2\varphi^*\varphi$

Then,

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial\varphi/\partial t)} = \frac{\partial\varphi^*}{\partial t} = \dot\varphi^* = \sqrt{1/2}\,(\dot\varphi_1 - i\dot\varphi_2)$$

$$\pi^* = \dot\varphi$$

or

$$\pi_{1,2} = \frac{\partial\mathcal{L}}{\partial(\partial\varphi_{1,2}/\partial t)} \Rightarrow \quad \begin{aligned}\pi &= \sqrt{1/2}\,(\pi_1 - i\pi_2)\\ \pi^* &= \sqrt{1/2}\,(\pi_1 + i\pi_2)\end{aligned}$$

So,

$$\mathcal{H} = \sum_{i=1}^{2} \pi_i\dot\varphi_i - \mathcal{L}$$

$$\begin{aligned}
&= \pi_1\dot\varphi_1 + \pi_2\dot\varphi_2 - \mathcal{L}\\
&= \sqrt{1/2}\,(\pi+\pi^*)\sqrt{1/2}\,(\dot\varphi + i\dot\varphi^*) + \sqrt{1/2}\,(\pi-\pi^*)\sqrt{1/2}\,(\dot\varphi - \dot\varphi^*) - \mathcal{L}\\
&= (1/2)\,(\pi\dot\varphi + \pi\dot\varphi^* + \pi^*\dot\varphi^* + \pi^*\dot\varphi^*)\\
&\quad + (1/2)\,(\pi\dot\varphi - \pi\dot\varphi^* - \pi^*\dot\varphi + \pi^*\dot\varphi^*) - \mathcal{L}\\
&= \pi\dot\varphi + \pi^*\dot\varphi^* - \mathcal{L}\\
&= \pi\pi^* + \pi^*\pi - \dot\varphi^*\dot\varphi + \vec\nabla\varphi^*\cdot\vec\nabla\varphi + m^2\varphi^*\varphi\\
&= \pi\pi^* + \pi^*\pi - \pi\pi^* + \vec\nabla\varphi^*\cdot\vec\nabla\varphi + m^2\varphi^*\varphi
\end{aligned}$$

$$\mathcal{H} = \pi^*\pi + \vec\nabla\varphi^*\cdot\vec\nabla\varphi + m^2\varphi^*\varphi \qquad \text{save} —$$

Everything from before w/ one-component φ holds here -- just add $\sum_{1,2}$ ...

Hamilton's principle (we're not varying δx^μ here)

$$\delta S = \int d^4x \left\{ \frac{\partial}{\partial x^\mu} \left[\sum_i \frac{\partial \mathcal{L}}{\partial(\partial\varphi_i/\partial x^\mu)} \delta\varphi_i \right] \right\} = 0$$

$$= \int d^4x \frac{\partial}{\partial x^\mu} \left\{ \frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial\varphi^*/\partial x^\mu)} \delta\varphi^* \right\} = 0$$

$$= i\alpha \int d^4x \frac{\partial}{\partial x^\mu} \left\{ -\frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)} \varphi + \frac{\partial \mathcal{L}}{\partial(\partial\varphi^*/\partial x^\mu)} \varphi^* \right\} = 0$$

and we identify.

$$J^\mu(x) = i \left\{ \frac{\partial \mathcal{L}}{\partial(\partial\varphi^*/\partial x^\mu)} \delta\varphi^* - \frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x^\mu)} \varphi \right\}$$

$$= i \left\{ \frac{\partial\varphi}{\partial x_\mu} \varphi^* - \frac{\partial\varphi^*}{\partial x_\mu} \varphi \right\} = [J^0, \vec{J}]$$

$$J^0 = i \left\{ \frac{\partial\varphi}{\partial t} \varphi^* - \frac{\partial\varphi^*}{\partial t} \varphi \right\} = +i [\pi^* \varphi^* - \pi\varphi]$$

$$J^j = i \left\{ -\frac{\partial\varphi}{\partial x^j} \varphi^* + \frac{\partial\varphi^*}{\partial x^j} \varphi \right\}$$

$$\vec{J} = i [-\varphi^* \vec{\nabla}\varphi + \varphi \vec{\nabla}\varphi^*]$$

at the surface, the ~~space~~ integral of space pieces $\to 0$, so

$$\delta S = i\alpha \int d^4x \frac{\partial}{\partial t} [\pi^* \varphi^* - \pi\varphi] = 0$$

So, $J^0 = \rho(x) = i[\pi^* \varphi^* - \pi \varphi]$ ← notice if real → $\rho(x)=0$ and so $Q=0$ and no symmetry.

$$\delta S = \alpha \int dt \frac{\partial}{\partial t} \int d^3x \; \rho(x)$$

$$= \alpha \int dt \frac{\partial Q}{\partial t}$$

So, a continuity equation is the result of conservation:

$$\vec{\nabla}\cdot\vec{J} = i\left\{ -\vec{\nabla}\varphi^* \cdot \vec{\nabla}\varphi + \vec{\nabla}\varphi \cdot \vec{\nabla}\varphi^* + \varphi \nabla^2 \varphi^* - \varphi^* \nabla^2 \varphi \right\}$$

$$= i\left[\varphi \nabla^2 \varphi^* - \varphi^* \nabla^2 \varphi\right]$$

$$\frac{\partial J^0}{\partial t} = i\left[\dot{\pi}^* \varphi^* - \dot{\pi}\varphi + \pi^* \dot{\varphi}^* - \pi \dot{\varphi}\right]$$

$$= i(\dot{\pi}^* \varphi^* - \dot{\pi}\varphi)$$

So,

$$\frac{\partial J^0}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$$

as can be shown from above plus K.G. equation.

Fermions: Same philosophy...

What $\mathcal{L}$ will give us the Dirac Equation?

$$\mathcal{L}(x) = \bar{\Psi}(x)\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\Psi(x) = \bar{\Psi}(x)[i\not{\partial} - m]\Psi(x)$$

does.

Put in matrix indices

$$\mathcal{L}(x) = \bar{\psi}_j(x)\left[i\gamma^\mu_{jk} \frac{\partial}{\partial x^\mu} - \cancel{}\,\delta_{jk}\, m \right] \psi_k(x)$$

μ: 0-3

j, k: 1-4

The E-L prescription:

$$\bar{\psi}(x)\left[i \frac{\overleftarrow{\partial}}{\partial x^\mu} \gamma^\mu + m \right] = 0$$

$$\left[i\gamma^\mu \frac{\partial}{\partial x^\mu} - m \right] \psi(x) = 0$$

The conjugate momenta are

$$\pi_j(x) = i\,\psi_j^+(x) \qquad \bar{\pi}_j(x) = 0$$

So, $\mathcal{H} = \psi^+(x)\left[\vec{\alpha}\cdot\vec{p} + \beta m \right] \psi(x)$

spin 1. --

The measurables for E & M are $\vec{E}$ and $\vec{B}$ which satisfy Maxwell's Equations

$$\text{(A)} \qquad \vec{\nabla}\cdot\vec{E} = \rho \qquad\qquad \vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{J}$$

$$\text{(B)} \qquad \vec{\nabla}\cdot\vec{B} = 0 \qquad\qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

The potentials are derived quantities and obtained via,

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad \text{and} \quad \vec{E} = -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t}$$

The Lagrangian formalism (Leonard 1900) --

A covariant formalism first:

presume

$$A^\mu = [\phi, \vec{A}] \quad \& \quad j^\mu = [\rho, \vec{J}]$$

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}A^0 - \frac{\partial\vec{A}}{\partial x^0} \quad ,$$

(remember $\quad \partial_\mu \equiv [\frac{\partial}{\partial t}, +\vec{\nabla}]$

$$E^j = -g^{jk}\frac{\partial A^0}{\partial x^k} - \frac{\partial A^j}{\partial x^0} = +\frac{\partial A^0}{\partial x_j} - \frac{\partial A^j}{\partial x_0} \equiv F^{0j}$$

and $\quad \vec{B} = \vec{\nabla}\times\vec{A}$: tensor curl:

$$B^i = \varepsilon^{jki}\frac{\partial A_j}{\partial x^k} = \varepsilon^{jki} g_{jq}\frac{\partial A^q}{\partial x^k}$$

For example

$$B^1 = \varepsilon^{jk1} g_{jq} \frac{\partial A^q}{\partial x^k}$$

$$= \varepsilon^{231} g_{22} \frac{\partial A^2}{\partial x^3} + \varepsilon^{321} g_{33} \frac{\partial A^3}{\partial x^2}$$

$$= (1)(-1)\frac{\partial A^2}{\partial x^3} + (-1)(-1)\frac{\partial A^3}{\partial x^2} = \frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3}$$

or $\quad B^i = \frac{\partial A^k}{\partial x^j} - \frac{\partial A^j}{\partial x^k} \qquad i, j, k$ cyclic.

$$= -\frac{\partial A^k}{\partial x_j} + \frac{\partial A^j}{\partial x_k} = \frac{\partial A^j}{\partial x_k} - \frac{\partial A^k}{\partial x_j} \equiv F^{jk}$$

These can be put together to make the "Field Strength Tensor"

$$F^{\mu\nu} = \frac{\partial A^\mu}{\partial x_\nu} - \frac{\partial A^\nu}{\partial x_\mu}$$ an antisymmetric tensor.

$$\begin{array}{c c} & 0 \longrightarrow 3 \\ \begin{array}{c} 0 \\ \downarrow \\ 3 \end{array} & \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix} \end{array}$$

or $\quad F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu$

Since F is antisymmetric, from

$$T^{\sigma\mu\nu} = \frac{\partial F^{\mu\nu}}{\partial x_\sigma} + \frac{\partial F^{\sigma\mu}}{\partial x_\nu} + \frac{\partial F^{\nu\sigma}}{\partial x_\mu}$$

$$= \frac{\partial}{\partial x_\sigma}\left(\partial^\nu A^\mu - \partial^\mu A^\nu\right) + \frac{\partial}{\partial x_\nu}\left(\partial^\mu A^\sigma - \partial^\sigma A^\mu\right) + \frac{\partial}{\partial x_\mu}\left(\partial^\sigma A^\nu - \partial^\nu A^\sigma\right)$$

cancel, etc.

so, $\quad T^{\sigma\mu\nu} = 0$

look at some:

$$T^{123} = 0 = \frac{\partial F^{23}}{\partial x_1} + \frac{\partial F^{12}}{\partial x_3} + \frac{\partial F^{31}}{\partial x_2}$$

$$= \frac{\partial B^1}{\partial x_1} + \frac{\partial B^3}{\partial x_3} + \frac{\partial B^2}{\partial x_2}$$

$$= -\left(\frac{\partial B^1}{\partial x^1} + \frac{\partial B^2}{\partial x^2} + \frac{\partial B^3}{\partial x^3}\right) = -\vec{\nabla}\cdot\vec{B}$$

so $\quad T^{123} = 0 \Rightarrow \vec{\nabla}\cdot\vec{B} = 0 \quad$ B

$$T^{021} = 0$$

$$= \cancel{\frac{\partial F^{21}}{\partial x_0}} \quad -\frac{\partial B^3}{\partial t} - \left(\vec{\nabla}\times\vec{E}\right)^3 = 0 \quad \text{B}$$

So, the B M.E. are satisfied by $T^{\sigma\mu\nu} = 0$
(sometimes see $\mathcal{F}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ and this becomes $\dfrac{\partial \mathcal{F}^{\mu\nu}}{\partial x^\mu} = 0$)

Look at $\frac{\partial F^{\mu\nu}}{\partial x^\nu}$

$\mu=0$: $$\frac{\partial F^{0\nu}}{\partial x^\nu} = \cancel{\frac{\partial F^{00}}{\partial x^0}}^{\,0} + \frac{\partial F^{01}}{\partial x^1} + \frac{\partial F^{02}}{\partial x^2} + \frac{\partial F^{03}}{\partial x^3}$$

$$= \vec{\nabla}\cdot\vec{E} \text{ which is } = \rho \qquad \text{A}$$

$\mu=i$ $$\frac{\partial F^{i\nu}}{\partial x^\nu} = \frac{\partial F^{i0}}{\partial x^0} + \frac{\partial F^{i1}}{\partial x^1} + \frac{\partial F^{i2}}{\partial x^2} + \frac{\partial F^{i3}}{\partial x^3}$$

in order to not be trivially zero, $\frac{\partial F^{ij}}{\partial x^j} \Rightarrow i \neq j$

if $i \neq 0 \Rightarrow B$ terms.

$$\frac{\partial F^{1\nu}}{\partial x^\nu} = \frac{\partial F^{10}}{\partial x^0} + \cancel{\frac{\partial F^{11}}{\partial x^1}}^{\,0} + \frac{\partial F^{12}}{\partial x^2} + \frac{\partial F^{13}}{\partial x^3}$$

$$= -\frac{\partial E^1}{\partial x^0} + 0 + \frac{\partial B^3}{\partial x^2} - \frac{\partial B^2}{\partial x^3}$$

$$= -\frac{\partial E^1}{\partial t} + (\vec{\nabla}\times\vec{B})^1$$

so $$\frac{\partial F^{1\nu}}{\partial x^\nu} = \left(-\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right)^1 = J^1 \qquad \text{A}$$

So, M.E. of both kinds are satisfied in covariant form by

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = j^\mu \quad \text{and} \quad T^{\sigma\mu\nu} = 0$$

Remember that the A^μ are ambiguous. There are gauge transformations that leave measureables unchanged.

A variety of gauges are used:

Coulomb gauge: $\vec{\nabla}\cdot\vec{A} = 0$
Lorentz gauge: $\partial_\mu A^\mu = 0$
Temporal gauge: $A_0 = 0$
and others.

Lorentz Gauge is covariant and the ambiguity is expressed by

$$A^\mu \to A^{\mu\prime} = A^\mu + \frac{\partial\theta(x)}{\partial x_\mu}$$

$\theta(x)$ - some scalar function of x.

Note: $$F'^{\mu\nu} = \frac{\partial A'^\mu}{\partial x_\nu} - \frac{\partial A'^\nu}{\partial x_\mu} = F^{\mu\nu}$$

~~This can be ... by ...~~

If $\theta(x)$ is choosen such that

$$\frac{\partial A^\mu}{\partial x^\mu} = -\frac{\partial^2\theta(x)}{\partial x^\mu \partial x_\nu} \to$$

then

$$\frac{\partial A'^\mu}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}\left(A^\mu + \frac{\partial\theta}{\partial x_\mu}\right)$$

$$= \frac{\partial A^\mu}{\partial x^\mu} + \frac{\partial^2\theta}{\partial x^\mu \partial x_\mu} = 0 ,$$

Then, M.E. (sourceless) give

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = 0$$

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial A^\mu}{\partial x_\nu} - \frac{\partial A^\nu}{\partial x_\mu}\right) = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x_\nu} - \frac{\partial A^\nu}{\partial x^\nu \partial x_\mu} = 0$$

$$= \frac{\partial^2 A}{\partial x^\nu \partial x_\nu} - \frac{\partial}{\partial x_\mu}\Big(\underbrace{\frac{\partial A^\nu}{\partial x^\nu}}_{\text{L.G.} \Rightarrow 0}\Big) = 0$$

and

$$\frac{\partial^2 A^\mu}{\partial x^\nu \partial x_\nu} = 0$$

$$\left.\begin{aligned}\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) A^\mu &= 0 \\ \Box A^\mu &= 0\end{aligned}\right\}$$ A Klein-Gordon wave equation for A^μ.

From above, including sources

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = j^\mu \quad \Rightarrow \quad \Box A^\mu = j^\mu$$

"source" of A^μ.

For this classical discussion, the correct $\mathcal{L}(x)$ is

$$\mathcal{L}(x) = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

which can be seen component by component...

$$F^{\mu\nu}F_{\mu\nu} = F^{\mu\nu}F^{\alpha\beta}g_{\alpha\mu}g_{\beta\nu}$$

$$= \underset{A}{F^{\mu 0}F^{\alpha 0}}g_{\alpha\mu}g_{00} + \underset{B}{F^{\mu 1}F^{\alpha 1}}g_{\alpha\mu}g_{11} + \underset{C}{F^{\mu 2}F^{\alpha 2}}g_{\alpha\mu}g_{22} + \underset{D}{F^{\mu 3}F^{\alpha 3}}g_{\alpha\mu}g_{33}$$

$$\text{A}\ \sum_\mu = \underset{\alpha=1}{F^{10}F^{\alpha 0}g_{\alpha 1}g_{00}} + \underset{\alpha=2}{F^{20}F^{20}g_{\alpha 2}g_{00}} + \underset{\alpha=3}{F^{30}F^{30}g_{33}g_{00}}$$

$$= (F^{10})^2(-1)(1) + (F^{20})^2(-1)(1) + (F^{30})^2(-1)(1)$$

$$= -E_1^2 - E_2^2 - E_3^2$$

$$\text{B}\ \sum_\mu = \underset{\alpha=0}{F^{01}F^{\alpha 1}g_{\alpha 0}g_{\phi\phi}} + \underset{\alpha=2}{F^{21}F^{\alpha 1}g_{\alpha 2}g_{11}} + \underset{\alpha=3}{F^{31}F^{\alpha 1}g_{\alpha 3}g_{11}}$$

$$= F^{01}F^{01}g_{00}g_{\phi\phi} + \dots$$

$$= -(F^{01})^2 + (F^{21})^2 + (F^{31})^2$$

$$= -E_1^2 + B_3^2 + B_2^2$$

etc C and D

$$\mathcal{L} = \tfrac{1}{2}(\vec{E}^2 - \vec{B}^2) \qquad \checkmark$$

Euler-Lagrange equations are complicated

For each β: $$\frac{\partial}{\partial x_\alpha}\left(\frac{\partial \mathcal{L}}{\partial(\partial A^\beta/\partial x_\alpha)}\right) - \frac{\partial \mathcal{L}}{\partial A^\beta} = 0$$

$$\frac{\partial \mathcal{L}}{\partial A^\beta} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial A^\beta/\partial x_\alpha)} = -\tfrac{1}{4}\left\{ \frac{\partial F^{\mu\nu}}{\partial(\partial A^\beta/\partial x_\alpha)} F_{\mu\nu} + F^{\mu\nu} \frac{\partial F_{\mu\nu}}{\partial(\partial A^\beta/\partial x_\alpha)} \right\}$$

From $F^{\mu\nu} = \dfrac{\partial A^\mu}{\partial x_\nu} - \dfrac{\partial A^\nu}{\partial x_\mu}$ or $F_{\mu\nu} = \dfrac{\partial A_\mu}{\partial x^\nu} - \dfrac{\partial A_\nu}{\partial x^\mu}$

$$\frac{\partial F^{\mu\nu}}{\partial(\partial A^\beta/\partial x_\alpha)} = \delta^\mu{}_\beta \delta^\nu{}_\alpha - \delta^\nu{}_\beta \delta^\mu{}_\alpha = \frac{\partial F^{\mu\nu}}{\partial(\partial^\alpha A^\beta)}$$

and

$$\frac{\partial F_{\mu\nu}}{\partial(\partial A^\beta/\partial x_\alpha)} = g_{\mu\eta} g_{\nu\rho} \frac{\partial F^{\eta\rho}}{\partial\left(\frac{\partial A^\beta}{\partial x_\alpha}\right)}$$

$$= g_{\mu\eta} g_{\nu\rho} \left(\delta^\eta{}_\beta \delta^\rho{}_\alpha - \delta^\rho{}_\beta \delta^\eta{}_\alpha \right)$$

$$= g_{\mu\beta} g_{\nu\alpha} - g_{\mu\alpha} g_{\nu\beta}$$

so...

$$\frac{\partial \mathcal{L}}{\partial(\partial A^\beta/\partial x_\alpha)} = -\frac{1}{4}\left\{ \left(\delta^\mu{}_\beta \delta^\nu{}_\alpha - \delta^\nu{}_\beta \delta^\mu{}_\alpha\right) F_{\mu\nu} + \left(g_{\mu\beta} g_{\nu\alpha} - g_{\mu\alpha} g_{\nu\beta}\right) F^{\mu\nu} \right\}$$

$$= -\frac{1}{4}\left\{ F_{\beta\alpha} - F_{\alpha\beta} + F_{\beta\alpha} - F_{\alpha\beta} \right\}$$

$$= -\frac{1}{4}\left\{ 2F_{\beta\alpha} - 2F_{\alpha\beta} \right\} = -\frac{1}{2}\left\{ F_{\beta\alpha} - F_{\alpha\beta} \right\}$$

$$= -F_{\alpha\beta} = F_{\beta\alpha}$$

So, the E-L equations are (for each A^β)

$$\frac{\partial F_{\beta\alpha}}{\partial x_\alpha} = 0$$ which give 2 M.E.'s in free space (A)

By working with A^μ and the definitions, we automatically get the (B) M.E.'s

$\mathcal{H}$:

$$\pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\dot{A}_\mu)} = \frac{\partial \mathcal{L}}{\partial(\partial A_\mu/\partial x^0)}$$

From above

$$\frac{\partial \mathcal{L}}{\partial(\partial A_\mu/\partial x^0)} = F^{\mu 0} \Rightarrow F^{i0} \qquad i \neq 0$$

So,
$$\pi^\mu(x) = \pi^i(x) = E^i$$

Then

$$\begin{aligned}
\mathcal{H}(x) &= \pi^\mu \dot{A}_\mu - \mathcal{L} \\
&= \pi^i \dot{A}_i - \mathcal{L} \\
&= \pi^i \frac{\partial A_i}{\partial x^0} - \mathcal{L} = -\pi^i \frac{\partial A^i}{\partial x^0} - \mathcal{L}
\end{aligned}$$

$$\frac{\partial A^i}{\partial x^0} = (-\vec{\nabla}\varphi)^i - (\vec{E})^i$$ from defining relation

$$\mathcal{H}(x) = -(E^i)(-E^i) + (E^i)\left(\frac{\partial \varphi}{\partial x_i}\right) - \tfrac{1}{2}E^2 + \tfrac{1}{2}B^2$$

$$= \tfrac{1}{2}E^2 + \tfrac{1}{2}B^2 + \vec{E}\cdot\vec{\nabla}\varphi.$$

Interactions of EM field... the modern way.

Imagine a global $U(1)$ transformation in Hilbert space

$$|\psi(x)\rangle \longrightarrow |\psi'(x)\rangle = R(\theta)|\psi(x)\rangle \cancel{= e^{iq\theta}|\psi(x)\rangle}$$

"global" means θ is <u>constant</u>.

$R(\theta) = e^{iQ\theta}$, which we presume can be built up from a series of infinitesimal rotations.

$$e^{iQ\theta}|\psi(x)\rangle \longrightarrow e^{iq\theta}|\psi(x)\rangle \quad \text{where} \quad Q|\psi(x)\rangle = |\psi(x)\rangle q.$$

So,

$$R(\theta)|\psi(x)\rangle = e^{iq\theta}|\psi(x)\rangle$$ a constant phase rotation

Look at D.E. $\quad (i\not\partial - m)\psi(x) = 0$

transform <u>it</u>

$$\begin{aligned}(i\not\partial - m)\psi(x) &\rightarrow (i\not\partial - m)\psi'(x)\\ &= (i\not\partial - m)e^{iq\theta}\psi(x) = 0\\ &= (i\not\partial - m)\psi(x)\end{aligned}$$

So, the same.

$\Rightarrow$ Dirac's theory is invariant wrt a global $U(1)$ transformation

The Noether charge associated with such a transformation has the properties of electric charge – or BARYON number or LEPTON number or HYPERCHARGE – all symmetries w/ single values for all ψs.

Now consider a local U(1) transformation $\Rightarrow$ ~~θt~~ $\theta = \theta(x)$

$$|\psi(x)\rangle \to |\psi'(x)\rangle = R[\theta(x)]\,|\psi(x)\rangle = e^{iq\theta(x)}|\psi(x)\rangle$$

where still $\quad Q|\psi\rangle = |\psi\rangle q \quad$ but the value depends upon spacetime – when and where the rotation occurs.

Dirac Equation:

$$(i\not\partial - m)\psi'(x) = 0$$
$$(i\not\partial - m)\, e^{iq\theta(x)}\psi(x) = 0$$
$$i\partial_\mu \gamma^\mu e^{iq\theta(x)}\psi(x) - m e^{iq\theta(x)}\psi(x) = 0$$
$$i\left\{ iq[\partial_\mu\theta(x)]\gamma^\mu e^{iq\theta(x)}\psi(x) + [\partial_\mu\psi]\, e^{iq\theta(x)}\gamma^\mu \right\} - m e^{iq\theta(x)}\psi(x) = 0$$
$$\left\{ -q[\partial_\mu\theta]\gamma^\mu + i\partial_\mu\gamma^\mu - m \right\}\psi(x) = 0$$

$$i(\not\partial - m)\psi(x) - q\underbrace{(\not\partial\,\theta(x))}_{\text{spoils the invariance}}\psi(x) = 0$$

Suppose you decide that an acceptable theory must be invariant wrt local U(1) transformations.

Then, something must be done with the gradient term.

<u>Define</u> "covariant derivative" $D^\mu \equiv \partial^\mu + X^\mu$

some 4-vector.

Determine X^μ such that

$$(i\not{\partial} - m)\psi(x) = 0 \longrightarrow (i\not{D} - m)\psi'(x) = 0$$

$$[i\gamma_\mu(\partial^\mu + X^\mu) - m]\, e^{iq\theta(x)}\psi(x) = 0$$

$$i\gamma_\mu\, iq[\partial^\mu\theta]\psi e^{iq\theta(x)} + i\gamma_\mu[\partial^\mu\psi]e^{iq\theta(x)} + i\gamma_\mu X^\mu\psi e^{iq\theta(x)} - m\psi e^{iq\theta(x)} = 0$$

$$-q\not{\partial}\theta(x)\psi + i\not{\partial}\psi + i\not{X}\psi - m\psi = 0$$

$$i\gamma_\mu[\partial^\mu + X^\mu + iq\partial^\mu\theta(x)]\psi - m\psi = 0$$

Still not right -- so D^μ definition must be accompanied by an arbitrariness in X^μ:

$$X^\mu \longrightarrow X^{\mu\prime} = X^\mu - iq\partial^\mu\theta(x) \quad \text{along w/ } e^{iq\theta(x)}\psi(x)$$

Then, $i\gamma_\mu[\partial^\mu + X^{\mu\prime} + iq\partial^\mu\theta]\psi - m\psi = 0$

$$i\gamma_\mu[\partial^\mu + X^\mu - iq\partial^\mu\theta + iq\partial^\mu\theta]\psi - m\psi = 0$$

$$i\gamma_\mu[\partial^\mu + X^\mu]\psi - m\psi = 0$$

$$(i\not{D} - m)\psi(x) = 0$$

Remember the time-honored way to add an electromagnetic field to a classical problem - The "minimum coupling rule" -

$$\vec{p} \to \vec{p} - q\vec{A} \quad \text{or} \quad p^\mu \to p^\mu - qA^\mu$$

when quantized

$E \xrightarrow{QI} i\frac{\partial}{\partial t}$

$\vec{p} \xrightarrow{QI} -i\vec{\nabla}$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + iq\phi$$

$$-\vec{\nabla} \to -\vec{\nabla} + iq\vec{A}$$

So, the rule says $\partial^\mu = [\frac{\partial}{\partial t}, -\vec{\nabla}] \to [\frac{\partial}{\partial t} + iq\phi, -\vec{\nabla} + iq\vec{A}]$

$\to \partial^\mu + iqA^\mu$

which we can identify as D^μ.

if: $iqA^\mu \equiv X^\mu$.

Then the subsidiary condition on X^μ to go along with $e^{iq\theta}\psi$ becomes

$$X^\mu \to X^\mu - iq\partial^\mu\theta$$

$$iqA^\mu \to iqA^\mu - iq\partial^\mu\theta$$

$$A^\mu \to A^\mu - \partial^\mu\theta$$

which is the Gauge Invariance

$\vec{A} \to \vec{A} + \vec{\nabla}\theta$

$\phi \to \phi - \frac{\partial\theta}{\partial t}$

So, the U(1) Gauge Invariant Dirac Equation is:

$$(i \not{D} - m)\,\psi(x) = 0$$

$$i \gamma_\mu (\partial^\mu + iqA^\mu)\psi - m\psi = 0$$

$$\left.(i\not{\partial} - m)\,\psi(x) = q \not{A}\,\psi(x)\right\} \text{ a source}$$

electric charge has entered via the demand of symmetry.

A^μ — a whole new dof – "particle" has entered via symmetry! The photon HAS TO EXIST in order to have local U(1) symmetry preserved.