phase - space

Let's deal with phase space separately, and then make use of it later.

Consider generally a two body process.
We could be speaking of a decay $a \to 3+4$ for which we would calculate the decay rate / particle

$$d\omega = \frac{(2\pi)^4 \delta(a \to 3+4)\, |T|^2\, dP_3\, dP_4}{2E_a}$$

and then the decay rate, "width" is

$$\Gamma = \int \frac{d\omega}{d\Omega}\, d\Omega$$

or the cross section for $1+2 \to 3+4$

$$d\sigma = \frac{(2\pi)^4 \delta(1+2 \to 3+4)\, |T|^2\, dP_3\, dP_4}{(\text{flux} \cdot \text{normalization})}$$

Common is the so-called phase space factor, which I'll represent as

$$d_2\rho(\vec{P}_3, \vec{P}_4) \equiv (2\pi)^4 \delta(P_1 + P_2 - P_3 - P_4)\, d\vec{P}_3\, d\vec{P}_4$$

We're going to assume that particle 3 will be measured and particle 4 will be ignored, so we need to "integrate 4 away".

Let's define

$$d_2\rho(\vec{p}_3) = d_2\rho(\Omega_3, p_3) \equiv \int d_2\rho(\vec{p}_3, \vec{p}_4)\, d^3\vec{p}_4$$

Then,

$$d_2\rho(\vec{p}_3) = \frac{1}{(2\pi)^2} \int \frac{d\Omega_3\, p_3^2\, dp_3}{4E_3E_4}\, d^3\vec{p}_4\, \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\,\delta(E_1+E_2-E_3-E_4)$$

Obviously, the $\delta^3(\,)$ function insures overall 3 momentum conservation

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4$$
$$\vec{p}_4 = \vec{p}_1 + \vec{p}_2 - \vec{p}_3$$

which acts as a constraint. Then,

$$d_2\rho(\vec{p}_3) = \frac{1}{(2\pi)^2} \frac{d\Omega_3\, p_3^2\, dp_3}{4E_3E_4}\, \delta(E_1+E_2-E_3-E_4) \quad w/$$

Since $E_4^2 = p_4^2 + m_4^2$ and $\vec{p}_4 = \vec{p}_1 + \vec{p}_2 - \vec{p}_3$ the δ function is

$$\delta\left(E_1+E_2-E_3-\sqrt{(\vec{p}_1+\vec{p}_2+\vec{p}_3)^2+m_4^2}\,\right)$$

call $\left.\begin{aligned} E_a &\equiv E_1+E_2 \quad \text{and} \\ \vec{p}_a &\equiv \vec{p}_1+\vec{p}_2 \end{aligned}\right\}$ makes the following applicable to 2 body decay $a \to 3+4$ as well as scattering. $1+2 \to 3+4$

then

From the standard δ function manipulation

$$\delta[f(x)] = \left|\frac{\partial f}{\partial x}\right|^{-1}_{x=x_r} \delta(x-x_r) \qquad \text{when} \quad f(x_r)=0$$

Here,

$$f(E_3) = E_a - E_3 - E_4 \Rightarrow \left|\frac{\partial f}{\partial E_3}\right| = \left|-1-\frac{\partial E_4}{\partial E_3}\right|$$

$$E_{3r} = E_a - E_4 \qquad\qquad = 1 + \frac{\partial E_4}{\partial E_3}$$

Now $\quad E_4^2 = (\vec{p}_a - \vec{p}_3)^2 + m_4^2$

$$2E_4 \frac{\partial E_4}{\partial E_3} = 2p_3 \frac{\partial p_3}{\partial E_3} - 2\,\vec{p}_a \cdot \hat{p}_3 \frac{\partial p_3}{\partial E_3}$$

since $\quad p_3^2 = E_3^2 - m_3^2$

$$2p_3 \frac{\partial p_3}{\partial E_3} = 2E_3 \,, \qquad \frac{\partial p_3}{\partial E_3} = \frac{E_3}{p_3}$$

so

$$2E_4 \frac{\partial E_4}{\partial E_3} = 2p_3 \frac{E_3}{p_3} - 2\,\vec{p}_a \cdot \hat{p}_3 \frac{E_3}{p_3}$$

$$\frac{\partial E_4}{\partial E_3} = \frac{E_3}{E_4} - \vec{p}_a \cdot \vec{p}_3 \frac{E_3}{E_4 p_3^2} = \frac{E_3 p_3^2 - \vec{p}_a \cdot \vec{p}_3 E_3}{E_4 p_3^2}$$

so

$$\frac{\partial f}{\partial E_3} = 1 + \frac{\partial E_4}{\partial E_3} = \frac{E_4 p_3^2 + E_3 p_3^2 - \vec{p}_a \cdot \vec{p}_3 E_3}{E_4 p_3^2}$$

$$= \frac{E_a p_3^2 - \vec{p}_a \cdot \vec{p}_3 E_3}{E_4 p_3^2}$$

X

so, $$d_2\rho(\vec{p}_3) = \frac{1}{(2\pi)^2} \frac{d\Omega_3\, p_3^2 dp_3}{4E_3E_4} \left(\frac{E_4 p_3^2}{E_a p_3^2 - \vec{p}_a\cdot\vec{p}_3 E_{3r}} \right) \delta(E_3 - E_{3r})$$

need integral over E_3

Now $$d^3\vec{p}_3 = p_3^2\, d\Omega_3\, dp_3$$
$$p_3 = \sqrt{E_3^2 - m_3^2}$$
$$dp_3 = \frac{1}{2}\cdot\frac{2E_3 dE_3}{p_3} \Rightarrow p_3 dp_3 = E_3 dE_3$$

so, $$d_2\rho(\vec{p}_3) = \frac{1}{(2\pi)^2} d\Omega_3 \frac{p_3 \cancel{E_3} dE_3}{4\cancel{E_3}\cancel{E_4}} \left(\frac{\cancel{E_4} p_3^2}{E_a p_3^2 - \vec{p}_a\cdot\vec{p}_3 E_{3r}} \right) \delta(E_3 - E_{3r})$$

What's E_{3r} ?

$$E_{3r} = E_a - \sqrt{(\vec{p}_a - \vec{p}_3)^2 + m_4^2}$$ so,

$$(\vec{p}_a - \vec{p}_3)^2 + m_4^2 = (E_a - E_{3r})^2 = E_a^2 + E_{3r}^2 - 2E_aE_{3r}$$
$$p_a^2 + p_3^2 - 2\vec{p}_a\cdot\vec{p}_3 + m_4^2 =$$
$$\underbrace{p_a^2 - E_a^2}_{-m_a^2} + \underbrace{p_3^2 - E_{3r}^2}_{-m_3^2} - 2\vec{p}_a\cdot\vec{p}_3 + m_4^2 = -2E_aE_{3r}$$

and
$$E_{3r} = \frac{m_a^2 + m_3^2 - m_4^2 + 2\vec{p}_a\cdot\vec{p}_3}{2E_a}$$

and define
$$d_2\rho(\Omega_3) = \int d_2\rho(\vec{p}_3)\, dp_3 = \int d_2\rho(\vec{p}_3) \frac{E_3}{p_3} dE_3$$

$$= \frac{1}{(2\pi)^2} d\Omega_3 \frac{p_3^3}{4} \frac{1}{E_a p_3^2 - \vec{p}_a\cdot\vec{p}_3 E_{3r}}$$ where E_{3r}

now we can choose a frame.

choose the frame in which $\vec{p}_a = 0$ ($\Rightarrow \vec{p}_1 = -\vec{p}_2$ if a scattering problem, .. the "center of momentum frame" ... the rest frame for a decay problem).

Then: $$E_{3r} = \frac{m_a^2 + m_3^2 - m_4^2}{2m_a}$$

Since, $$p_3^2 = E_{3r}^2 - m_3^2$$
$$= \frac{m_a^4 + m_3^4 + m_4^2 + 2m_a^2 m_3^2 - 2m_a^2 m_4^2 - 2m_3^2 m_4^2}{4m_a^2} - m_3^2$$

$$= \frac{m_a^4 + m_3^4 + m_4^4 - 2m_a^2 m_3^2 - 2m_a^2 m_4^2 - 2m_3^2 m_4^2}{4m_a^2}$$

This numerator happens a lot in kinematics and is given a name —

the triangle function: $\lambda(a,b,c) \equiv a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$

so here we have

$$p_3^2 = \frac{\lambda(m_a^2, m_3^2, m_4^2)}{4m_a^2}$$

There is more jargon. The center of mass energy squared is called

$$s \equiv (p_1 + p_2)^2 = (p_3 + p_4)^2 = p_a^2 \qquad (4\text{ vectors})$$

$$s = m_a^2$$

so, you sometimes see (Jackson),

$$\left.\begin{aligned} E_3 &= \frac{s + m_3^2 - m_4^2}{2\sqrt{s}} \\ &\text{and you could also find} \\ E_4 &= \frac{s + m_4^2 - m_3^2}{2\sqrt{s}} \end{aligned}\right\} \text{ cm frame.}$$

also,

$$\vec{p}_3^{\,2} = \frac{[s - (m_3 + m_4)^2][s - (m_3 - m_4)^2]}{4s} = \vec{p}_4^{\,2}$$

and

$$|\vec{p}_3| = \frac{1}{2s} \lambda^{\frac{1}{2}}(s, m_3^2, m_4^2)$$

In this frame,

$$d_2\rho(\Omega_3) = \frac{1}{(2\pi)^2} \frac{p_3^3}{4} \frac{1}{m_a p_3^2} \, d\Omega_3$$

$$d_2\rho(\Omega) = \frac{p_3}{16\pi^2\sqrt{s}} \, d\Omega_3$$

and $$dw = \frac{|T|^2}{2E_a}\, d_2\rho(\Omega_3)$$

$$dw = \frac{|T|^2}{32\pi^2}\left(\frac{p_3}{m_a^2}\right) d\Omega_3$$

note $|T|^2$ will have θ_3 angular dependence, so this is as far as we can go

When integrated, this is the "total width" of the ~~unstable~~ A.

$$\Gamma \equiv \int \frac{dw}{d\Omega}\, d\Omega$$

The decay rate per particle is

$$\Gamma = -\frac{dN_A}{dt}\frac{1}{N_A}$$

so,

$$N_A(t) = N_A(0)\, e^{-\Gamma t}$$

and we refer to $\Gamma^{-1} \equiv \tau$ as the "lifetime"
(more later)

Now look at 2 body scattering, $1+2 \to 3+4$

$$d\sigma = \frac{|T|^2 d\phi_2}{(\text{flux}\cdot\text{normalization})}$$

$$\begin{aligned}(\text{flux}\cdot\text{normalization}) &= |\vec{v}_1 - \vec{v}_2|\, 2E_1 2E_2 \\ &= |\vec{\beta}_1 - \vec{\beta}_2|\, 4E_1E_2 \\ &= \left|\frac{\vec{p}_1}{E_1} - \frac{\vec{p}_2}{E_2}\right| 4E_1E_2 \\ &= 4\,|\vec{p}_1E_2 - \vec{p}_2E_1| \\ &= 4\sqrt{E_2^2|\vec{p}_1|^2 + E_1^2|\vec{p}_2|^2 - 2E_1E_2\,\vec{p}_1\cdot\vec{p}_2}\end{aligned}$$

which is general.

consider specific cases:

$$\xrightarrow{\vec{p}_1} \xleftarrow{\vec{p}_2}$$

but $|\vec{p}_1| \neq |\vec{p}_2|$ necessarily

i) anti colinear beams : $\vec{p}_1 \cdot \vec{p}_2 = -p_1 p_2$

$$(\text{flux}\cdot\text{norm}) = 4\sqrt{E_2^2 p_1^2 + E_1^2 p_2^2 + 2E_1E_2 p_1 p_2}$$

simplify $\quad 2E_1E_2 p_1 p_2 = (E_1E_2 + p_1 p_2)^2 - E_1^2E_2^2 - p_1^2 p_2^2$

$$\vdots$$

$$(\text{flux}\cdot\text{norm}) = 4\sqrt{-E_1^2 m_2^2 + p_1^2 m_2^2 + (E_1E_2 + p_1 p_2)^2}$$

write using invariants $\quad p_1^\mu \cdot p_{2\mu} = E_1E_2 - \vec{p}_1\cdot\vec{p}_2$
$\quad = E_1E_2 + |\vec{p}_1||\vec{p}_2|$

so

$$(\text{flux}\cdot\text{norm}) = 4\sqrt{(p_1\cdot p_2)^2 - m_1 m_2}$$

ii) center of momentum frame:

$$|\vec{p}_1| = |\vec{p}_2| \equiv p$$
$$|\vec{p}_3| = |\vec{p}_4| \equiv p'$$

then, $\quad (p_1\cdot p_2)^2 - m_1^2 m_2^2 = (E_1E_2 - \vec{p}_1\cdot\vec{p}_2)^2 - m_1^2 m_2^2$
$\quad = (E_1E_2 + |\vec{p}|^2)^2$
$\quad = p^2(E_1+E_2)^2$

so

$$(\text{flux}\cdot\text{norm}) = 4p(E_1+E_2)$$

since, for anti colinear beams,

$$s \equiv (p_1+p_2)^2 \qquad (4\text{ vectors})$$
$$= p_1^2 + p_2^2 + 2p_1\cdot p_2$$
$$= m_1^2 + m_2^2 + 2E_1E_2 - 2\vec{p}_1\cdot\vec{p}_2 \qquad \text{general}$$

$$= m_1^2 + m_2^2 + 2E_1E_2 + 2|\vec{p}|^2 \qquad \text{anti colinear \& center of momentum}$$

$$= p_1^2 + m_1^2 + p_2^2 + m_2^2 + 2E_1E_2$$
$$= E_1^2 + E_2^2 + 2E_1E_2$$
$$= (E_1+E_2)^2$$

obviously, if beams are same species (particle - particle or particle - antiparticle) then $E_1 = E_2 = E$

$$s = (2E)^2 \qquad \text{anticolinear, center of mom, same mass.}$$

and one typically refers to an accelerator's capability in terms of "root s", here

$\sqrt{s} = 2E$ @ Fermilab $E \simeq p = 1\ \text{TeV}$

$\sqrt{s} = 2\ \text{TeV}$

max: @ LEP $E \simeq p = 105\ \text{GeV}$

$\sqrt{s} = 210\ \text{GeV}$

@ LHC $E \simeq p = 7\ \text{TeV}$

$\sqrt{s} = 14\ \text{TeV}$

iii) For scattering in the stationary target frame (measure particle 3)

$$d\sigma(\vec{p}_3,\vec{p}_4)\ \frac{1}{\text{flux.norm}}\ (2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\ \Sigma|T|^2\ \frac{d^3p_3}{(2\pi)^3 2E_3}\ \frac{d^3p_4}{(2\pi)^3 2E_4}$$

do standard trick... $\frac{d^3p_4}{2E_4} \longrightarrow d^4p_4\ \delta(p_4^2-m_3^2)\,\theta(p_{04})$

and integrate, enforcing a constraint on 4-momentum conservation.

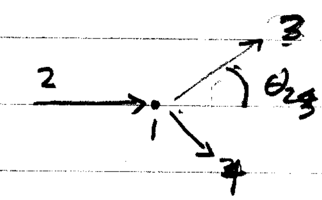
$$d\sigma(\vec{p}_3) = \frac{1}{\text{flux.norm}}\ \frac{1}{(2\pi)^2}\ \delta(p_4^2-m_4^2)\,\theta(p_{03})\,\frac{d^3p_3}{2E_3}\Bigg|_{p_4=p_1+p_2-p_3}$$

look at argument:

$$\begin{aligned} p_4^2-m_4^2 &= (p_1+p_2-p_3)^2-m_4^2 \\ &= m_1^2+m_2^2+m_3^2+2p_1\cdot p_2-2p_1\cdot p_3-2p_2\cdot p_3-m_4^2 \\ &= \underbrace{(m_1^2-m_4^2)}_{\equiv\, M_{14}^-}+\underbrace{(m_2^2+m_3^2)}_{\equiv\, M_{23}^+} \end{aligned}$$

in the lab, $p_1 = (m_1,\vec{0})$

$p_i = (E_i, \vec{p}_i)$



so,

$$\begin{aligned} p_4^2-m_4^2 &= M_{14}^- + M_{23}^+ + 2m_1E_2 - 2m_1E_3 - 2E_2E_3 + 2\vec{p}_2\cdot\vec{p}_3 \\ &= M_{14}^- + M_{23}^+ + 2m_1E_2 - 2E_3(m_1+E_2) + 2P_2P_4\cos\theta_{23} \\ &\equiv f(E_3) \end{aligned}$$

use standard δ function theorem

we'll convert to dE_3 – always can do this.

$$d^3p_3 = p_3^2\,dp_3\,d\Omega_3 \qquad p_3 = \sqrt{E_3^2 - m_3^2}$$

$$dp_3 = \tfrac{1}{2}\,\frac{2E_3}{p_3}\,dE_3$$

$$p_3\,dp_3 = E_3\,dE_3$$

call the root E_r, so $f(E_r) = 0 \rightarrow$ ugly

$$E_r = (M_{13}^- + M_{23}^+ + 2m_1E_2 + 2p_2p_3\cos\theta_{23})/2(m_1+E_2)$$

$$\frac{df}{dE_3} = 2p_2\cos\theta_{23}\,\underbrace{\frac{dp_3}{dE_3}}_{=\,E_3/p_3} - 2(m_1+E_2)$$

$$= 2p_2\cos\theta_{23}\,E_3/p_3 - 2(m_1+E_2)$$

so,

$$d\sigma(\Omega_3) = \frac{1}{\text{flux}\cdot\text{norm}}\,\frac{1}{(2\pi)^2}\int \frac{\delta(E_3-E_r)\,p_3\cancel{E_3}\,dE_3}{2\left(m_1+E_2-\frac{p_2}{p_3}E_3\cos\theta_{23}\right)}\,\frac{d\Omega_3}{2\cancel{E_3}}$$

$$\text{flux}\cdot\text{norm} = 4\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2} \qquad (4d)$$

$$= 4\sqrt{m_1^2E_2^2 - m_1^2m_2^2}$$

$$= 4m_1\sqrt{E_2^2 - m_2^2}$$

$$= 4m_1|\vec{p}_2|$$

so,

$$\frac{d\sigma}{d\Omega_3} = \frac{1}{4m_1|\vec{p}_2|}\,\frac{1}{(2\pi)^2}\,\frac{p_3}{4\left[m_1+E_2-\frac{p_2}{p_3}E_3\cos\theta_{23}\right]}\,\Sigma|T|^2\Bigg|_{E_3=E_r}$$

Unstable particles -- $A \to 1+2$

One imagines $N_A(0)$ particles @ $t=0$ with some average lifetime τ, such that

$$N_A(t) = N_A(0)\, e^{-t/\tau}$$

In accordance with the Uncertainty Principle, there is an uncertainty in the time each state exists which in turn leads to an uncertainty in the energy level according to $\Delta E \Delta t \geq \hbar$

The number of particles at any time would be related to $|\psi_A(t)|^2$. For free propagation

$$\psi_A(t) \propto \psi_A(0)\, e^{-iEt}.$$

That the state is unstable means,

$$|\psi_A(t)|^2 \propto |\psi_A(0)|^2 e^{-t/\tau}$$

Notice one gets this if one makes the energy imaginary

$$E = E_0 - \frac{i\Gamma}{2}$$

$$\psi_A(t) \sim \psi_A(0)\, e^{-iE_0 t}\, e^{-\Gamma/2\, t}$$

so

$$|\psi_A(t)|^2 \sim |\psi_A(0)|^2 e^{-\Gamma t}$$

Showing that $\frac{1}{\tau} = 2\,\mathrm{Im}(E) = \Gamma$

This constant is the decay rate, Γ

We always calculate in momentum space, so

$$\tilde{\psi}_A(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{iEt}\, \psi_A(t)$$

$$= \frac{1}{\sqrt{2\pi}} \psi_A(0) \int dt\, e^{i(E-E_0)t - \Gamma t/2}$$

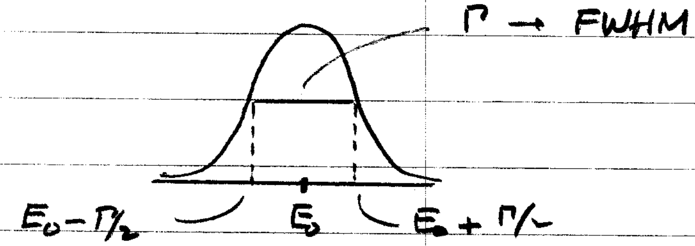
$$= \frac{i\,\psi_A(0)}{\sqrt{2\pi}} \frac{1}{(E-E_0) + i\Gamma/2}$$

The probability of finding A at this time.

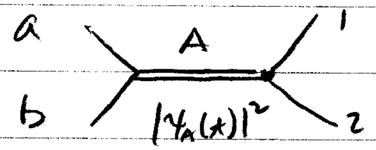
$$|\tilde{\psi}_A(E)|^2 \propto \frac{|\psi_A(0)|^2}{2\pi} \left(\frac{1}{(E-E_0)^2 + \Gamma^2/4} \right)$$

Breit Wigner shape

So, Γ is both the rate at which the states decay and the uncertainty in the value of the energy due to the finite lifetime



Often, an unstable state is produced in a scattering process $a + b \to A \to 1+2$



said to be a resonant state

The relativistic point of view is that the energy of the resonant peak is the mass of the resonance — definition.

$$E_0^2 = m_A^2 = s$$

write,

$$\frac{m_A}{m_A} \cdot \frac{1}{(E-E_0) + i\Gamma/2} = \frac{2m_A}{2m_A(E-E_0) + im_A\Gamma}$$

If $|E - m_A| \ll m_A$ — hard to think of a resonant "particle" interpretation otherwise

$$E^2 - m_A^2 = (E-m_A)(E+m_A)$$
$$\sim (E-m_A)\,2m_A$$

$$\frac{2m_A}{(E^2 - m_A^2) - im_A\Gamma}$$

which squared is proportional to,

$$\frac{1}{(s - m_A^2)^2 + m_A^2\Gamma^2}$$ Relativistic Breit-Wigner

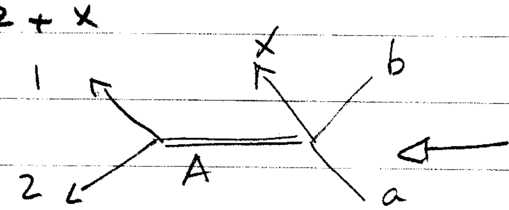
notice: $\dfrac{1}{E^2 - m_A^2 - i m_A \Gamma} \sim \dfrac{1}{E^2 - m_A^2 - i\eta}$

like the propagator for the particle which is intermediate. This leads to the definition of mass as the pole of the propagator.

The total amplitude for the process

$$a + b \to A \to 1 + 2 + x$$

can be written,



$$T(a+b \to 1+2+x) = T(A \to 1+2)\,\frac{1}{p^2 - m^2 - im\Gamma}\,T(a+b \to A+x)$$

imagine one intermediate state $a+b \to A \to 1+2$ then

$$\sigma(a+b \to 1+2) \propto \Gamma(A \to 1+2)\,\frac{1}{(s-m_A^2)^2 + m_A^2\Gamma^2}\,\Gamma(\bar{A} \to \bar{a}+\bar{b})$$

For calculational purposes, often the "narrow width approximation" is helpful.

$$\frac{1}{(s-m^2)^2 + m^2\Gamma^2} \sim \frac{\pi}{m\Gamma}\,\delta(s-m^2)$$