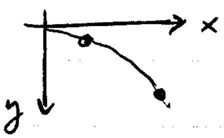
Principle of Least Action. – honorable, if confused, history:

- Heron – 4th century B.C.: attempted to explain light reflection by suggesting that light always travels in the shortest path.
- Fermat – 1657: modified this to be shortest time and then went on to use it to explain refraction in media of different indices. "Fermat's Principle"
- Newton – 1696: while at the mint, overnight met Johann Bernoulli's challenge to find the shape of the curve joining 2 points in gravitational field that would cause an object to "fall" in the least time. (also solved by Leibnitz, L'Hospital, and Jacob B.)

  → brachistochrone



- Euler – 1728: added friction – other problems & started the effort (largely geometrical, not analytic) to generalize the approach → known to be a problem of minimization
- Pierre-Louis Moreau de Maupertuis – 1744: generalized the Principle → a Unifying Principle, determined to explain why Nature always acts in simplest way → mechanics, optics, biology (!), proof of God

  Action ≡ velocity × distance

and sometimes Action ≡ $m \times (\text{velocity})^2$

For simple collisions $O \to^{v_1}$ $O \to^{v_2}$ $\Rightarrow$ $OO \xrightarrow{v}$

$S = m_1(v_1 - v)^2 + m_2(v - v_2)^2$ gives right v for

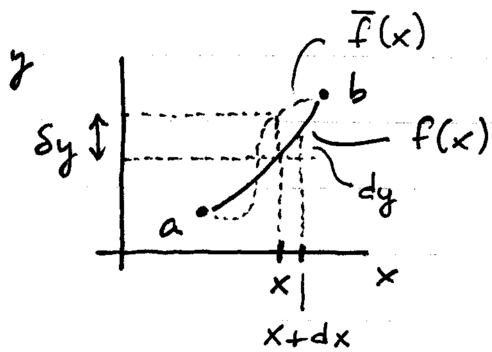
$S = \min \int 2T\,dt = \min \int mv\,ds$

Maupertius did nothing of any technical value —
but, he influenced Euler
and, was Lagrange's first patron

at age 23 Giuseppe Ladovico Lagrangia (1775) wrote to Euler (in Latin) of the first promising systematic approach ... a new kind of calculus.

Consider $$I = \int_a^b F[f(x), f'(x)]\,dx$$

and evaluate it "at" different functional forms (not points) ~~[illegible]~~ These different functions can be infinitesimally close to one another.



$$\bar{f}(x) = f(x) + \epsilon \eta(x) \quad *$$

infinitesimal parameter.

$$\bar{f}(x) - f(x) = \epsilon \eta(x) \equiv \delta f$$

an arbitrary, infinitesimal change in f at point x → the Variation

So: $x \to x + dx$ induces a change in $f \to f + df$ BUT δf is a brand new function

Formally, if $f(x)$ is a function in a Banach space (space of normed, linear functions):
$M = \{ f(x) ; x \in \mathbb{R} \}$ Then, a <u>Functional</u>, $F[f(x)]$

is a mapping $F: M \to \mathbb{R}$ or $\mathbb{C}$.

The variation — rate of change — of $F[f]$ with respect to change in $f(x)$ <u>at x</u> is

$$\frac{\delta F[f]}{\delta f(x)}$$ and is defined through

$$\delta F[f] = \int dx \underbrace{\frac{\delta F[f]}{\delta f(x)} \delta f(x)}_{} \qquad *$$

a superposition of local, range changes summed over x.

The Lagrange idea is to calculate the minimum of an integral, I, by finding the functional forms which make I minimum.

$$I = \int_a^b F[f, df/dx]\, dx \qquad *$$

$$\delta I = \delta \int_a^b F[f, f']\, dx = \int_a^b \bar{F}[f]\, dx - \int_a^b F[f]\, dx$$

$$= \int_a^b \bar{F} - F\, dx$$

$$= \int_a^b \delta F[f, f']\, dx \qquad *$$

$$\delta F[f,f'] = \int dx \left[\frac{\delta F}{\delta f(x)}\, \delta f(x) + \frac{\delta F}{\delta f'(x)}\, \delta f'(x) \right] \quad *$$

from $\star$, for now $F[\,f(x), df/dx\,]$

$$\left.\begin{array}{l} \delta f(x) \equiv \epsilon \eta(x) \\ \delta f'(x) \equiv \epsilon \eta'(x) \end{array}\right\} \quad \left(\begin{array}{l} \frac{d}{dx}\delta f = \frac{d}{dx}[\bar{f} - f] = \frac{d}{dx}[\epsilon\eta] = \epsilon\eta'(x) \\ \delta \frac{df}{dx} = \bar{f}' - f' = \epsilon \eta'(x) \end{array} \right.$$

$$\delta F = \epsilon \int_a^b \left(\frac{\delta F}{\delta f}\, \eta(x) + \frac{\delta F}{\delta f'}\, \eta'(x) \right) dx$$

All of the properties of derivatives hold -- and a parts-integration can be done on the second term

$$\int_a^b \frac{\delta F}{\delta f'}\, \eta'\, dx = \left. \frac{\delta F}{\delta f'}\, \eta \right|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\delta F}{\delta f'} \right) \eta\, dx$$

The function is equivalent to its varied form at the end points, so $\eta(a) = \eta(b) = 0$ and

$$\delta F[f,f'] = \epsilon \int_a^b \left[\frac{\delta F}{\delta f(x)} - \frac{d}{dx}\left(\frac{\delta F}{\delta f'(x)} \right) \right] \eta(x)\, dx \quad *$$

for arbitrary $\eta(x)$, or $\delta f(x) = \epsilon \eta(x)$,

$$\frac{\delta F}{\delta f(x)} - \frac{d}{dx}\left(\frac{\delta F}{\delta f'(x)} \right) = 0 \quad \text{for} \quad \delta I = 0 \quad *$$

which is now called the Euler-Lagrange Equation.

- Hamilton – 1834: imagined a "bundle" of light rays to be analogous to paths of mechanical particles.

for him... η (optics) $\sim$ v (mechanics)

everything of Hamilton is obscure...!

He defined something called the "Characteristic Function"

$$C \equiv \int \eta \, dx = \int \frac{c}{ds/dt} \, ds = c \int dt.$$

$\rightarrow$ Fermat's Principle $\delta C = 0$.

Further, he defined "Principle Function"

total energy, constant.

$$S \equiv C - Ht$$

w/ his analogy,

$$\begin{aligned} C &= m \int v \, ds \\ &= m \int \frac{ds}{dt} \frac{ds}{dt} \, dt = m \int v^2 dt. \\ &= \int 2T \, dt. \end{aligned}$$

So,

$$\begin{aligned} S &= \int 2T dt - \int H dt \\ &= \int [2T - (T + U)] dt \\ &= \int (T - U) \, dt \end{aligned}$$

$\delta S = 0$ is called Hamilton's Principle and $(T-U)$ is the "Free Energy" (Helmholtz) = "Lagrangian".

So, from the general introduction

$$\left.\begin{array}{lll} F \to L & & \\ x \to t & \text{the dependent variable} & \\ f\ y \to x & x = x(t) & \end{array}\right\}$$

$$\frac{\delta L}{\delta x(t)} - \frac{d}{dt}\frac{\delta L}{\delta \dot{x}(t)} = 0$$ follows, for one-particle dynamics.

For conservative systems: $\left.\begin{array}{l} U = V(x) \\ T = T(\dot{x}) \end{array}\right\}$ only

$$\frac{\partial L}{\partial x} = \frac{\partial U}{\partial x}$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}}$$ so E-L equation

$$\frac{\partial V}{\partial x} - \frac{d}{dt}\frac{\partial T}{\partial \dot{x}} = 0$$

Simple: $T = \frac{1}{2} m \dot{x}^2$ $\quad L = T - V = \frac{1}{2} m \dot{x}^2 - U(x)$

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x} = p$$

$$\frac{\partial V}{\partial x} - m\ddot{x} = 0 \Rightarrow m\ddot{x} = -\frac{\partial V}{\partial x} = f$$ Newton's 2nd Law!

2 Generalizations can follow:

1) non-conservative forces -- not in this course

2) "generalized coordinates" -- an idea of Lagrange ✓

and

$$\frac{\delta L}{\delta x} - \frac{\partial}{\partial t}\frac{\delta L}{\delta \dot{x}} = 0$$

$$\frac{\partial U}{\partial x} - m\ddot{x} = 0$$

$$m\ddot{x} = -\frac{\partial U}{\partial x} = f$$

2 Generalizations follow:

1) non-conservative forces -- not in this course

2) "generalized coordinates" -- an idea of Lagrange ✓

Generalized coordinates.

For N particles with no constraints, a system can be described by 3N, 3d coordinates, $\vec{X}$. If, however, there are constraints which in effect couple some coordinates together, then they are not independent. If these constraints can be expressed as a set of k equations of the form

$$f(x_1^1, x_2^1 x_3^1; x_1^2 x_2^2 x_3^2; \cdots x_1^N x_2^N x_3^N; t) = 0$$

then the constraints are "holonomic" and another set of ~~equation~~ parameters can be used to independently describe the system. There are 3N-k sets of these generalized coordinates

$$\left.\begin{aligned} \vec{X}_1 &= \vec{X}_1(q_1, q_2, q_3 \cdots q_{3N-k}, t) \\ \vec{X}_2 &= \vec{X}_2(q_1, q_2 \cdots q_{3N-k}, t) \\ &\vdots \\ \vec{X}_N &= \vec{X}_N(q_1, q_2 \cdots q_{3N-k}, t) \end{aligned}\right\} \quad \begin{aligned} x_{\rho,i} &= x_{\rho,i}(q_1 \cdots q_{3N-k}, t) \\ 1 \le \rho &\le N \end{aligned}$$

There may not be the 3-fold vector relationships among the q's.

Alternatively, the generalized coordinates can be expressed in terms of the originals

$$q_m = q_m(x_1^1, x_1^2, x_1^3 \, x \cdots x_1^N, x_2^1, x_2^2, x_2^3 \cdots x_2^N, x_3^1 \cdots x_3^N, t)$$

$$1 \le m \le N-k$$

Holding t fixed, a small displacement

$$\delta x_i = \frac{\partial x_i}{\partial q_m} \delta q_m$$

If Jacobian is $\neq 0$ -- $\left| \frac{\partial x_i}{\partial q_m} \right| \neq 0$, and this can be inverted

then $\delta q_m = \frac{\partial q_m}{\partial x_i} \delta x_i$ and we can integrate to get the q's as functions of x.

Phase space requires a derivative of x.-

$$dx = \frac{\partial x}{\partial q} dq + \frac{\partial x}{\partial t} dt \quad \text{so}$$

$$\dot{x}_{\rho i} = \frac{\partial x_{\rho, i}}{\partial q_m} q_m + \underbrace{\frac{\partial x_{\rho, i}}{\partial t}}_{= 0 \text{ if no explicit } t \text{ dependence.}} \qquad ☆$$

and

$$\ddot{x}_{\rho i} = \frac{\partial x_{\rho i}}{\partial q_m} \ddot{q}_m + \frac{\partial^2 x_{\rho i}}{\partial q_m \partial q_n} \dot{q}_m \dot{q}_n + \overbrace{2 \frac{\partial^2 x_{\rho i}}{\partial q_m \partial t} \dot{q}_m}^{0} + \frac{\partial^2 x_{\rho i}}{\partial t^2}$$

If q_m and $\dot{q}_m$ are independent

$$\cancel{\partial q} \quad \frac{\partial \dot{x}_{\rho i}}{\partial \dot{q}_m} = \frac{\partial x_{\rho i}}{\partial q_m}$$

The equations of motion are $F_{\rho i} = m_\rho \ddot{x}_{\rho i}$

Calculate work done by $F_{\rho i}$ over displacement $dx_{\rho i}$

$$W = \sum_{\rho i} F_{\rho i}\, dx_{\rho i}$$

$$= \sum_{\rho i} m_\rho \ddot{x}_{\rho i}\, dx_{\rho i} = \sum_{\rho i} m_\rho \ddot{x}_{\rho i} \left(\frac{\partial x_{\rho i}}{\partial q_m} dq_m + \frac{\partial x_{\rho i}}{\partial t} dt \right) \quad \text{A}$$

$$= \sum_{\rho i} m_\rho \left[\frac{d}{dt}\left(\dot{x}_{\rho i} \frac{\partial \dot{x}_{\rho i}}{\partial \dot{q}_m} \right) - \dot{x}_{\rho i} \frac{d}{dt}\left(\frac{\partial x_{\rho i}}{\partial q_m} \right) \right] dq_m$$

$$+ \sum_{\rho i} m_\rho \ddot{x}_{\rho i} \frac{\partial x_{\rho i}}{\partial t}\, dt$$

$$= \sum_{\rho i} \left[\frac{d}{dt} \frac{\partial}{\partial \dot{q}_m} \overbrace{\left(\tfrac{1}{2} m_\rho \dot{x}_{\rho i} \dot{x}_{\rho i} \right)}^{T} - \frac{\partial}{\partial q_m} \overbrace{\left(\tfrac{1}{2} m_\rho \dot{x}_{\rho i} \dot{x}_{\rho i} \right)}^{T} \right] dq_m$$

$$+ \sum_{\rho i} m_\rho \ddot{x}_{\rho i} \frac{\partial x_{\rho i}}{\partial t}\, dt$$

~~Apply Work Energy theorem $\Delta W = \Delta T$~~

$$W = \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_m} - \frac{\partial}{\partial q_m} T \right) dq_m + \sum m_\rho \ddot{x}_{\rho i} \frac{\partial x_{\rho i}}{\partial t}\, dt \qquad (1)$$

Define $\quad Q_m \equiv \sum_{\rho i} F_{\rho i} \frac{\partial x_{\rho i}}{\partial q_m} \qquad Q_t = \sum F_{\rho i} \frac{\partial x_{\rho i}}{\partial t}$

as "generalized force components" and the original expression for W can be written (A)

$$W = Q_m\, dq_m + Q_t\, dt \qquad (2)$$

equating ① = ②

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_m} - \frac{\partial T}{\partial q_m} = Q_m$$

Euler-Lagrange in terms of unconstrained applied forces.

If: no constraint involves time
no moving coordinate system

$\left.\right\}$ $T = T(\dot{q}_m)$ so $\frac{\partial x}{\partial t} = 0$

so

$$\dot{x}_{pi} = \frac{\partial x_{pi}}{\partial q_m}\dot{q}_m$$

$$T = \frac{1}{2}\sum_{pi} m_p (\dot{x}_{pi})^2 = \frac{1}{2}\sum_{p,i,m,n} m_p \frac{\partial x_{pi}}{\partial q_m}\frac{\partial x_{pi}}{\partial q_n}\dot{q}_m\dot{q}_n$$

If: forces are conservative, F_{pi} or Q_m, then

$$F_{pi} = -\frac{\partial U}{\partial x_{pi}}$$

so

$$Q_m \underset{\text{def}}{=} \sum F_{pi}\frac{\partial x_{pi}}{\partial q_m} = -\sum_{pi}\frac{\partial U}{\partial x_{pi}}\frac{\partial x_{pi}}{\partial q_m}$$

$$Q_m = -\frac{\partial U}{\partial q_m}$$

and E-L eq:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_m} - \frac{\partial T}{\partial q_m} + \frac{\partial U}{\partial q_m} = 0 = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_m} - \frac{\partial}{\partial q_m}(T-U) = 0$$

back to $L \equiv T-U$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m} - \frac{\partial L}{\partial q_m} = 0$$

$\left.\right\}$ one equation for each ~~dof~~ independent (generalized) dof

Just as for Cartesian coordinates

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} = p.$$

for generalized coordinates <u>define</u>

$$\text{"generalized momentum"} \equiv p_m \equiv \frac{\partial L}{\partial \dot{q}_m} \neq p.$$

For a particular system w/ conservative forces, consider

$$dL = (\partial L/\partial q_m)\, dq_m + (\partial L/\partial \dot{q}_m)\, d\dot{q}_m$$

$$\frac{dL}{dt} = \sum_m \frac{\partial L}{\partial q_m}\frac{dq_m}{dt} + \frac{\partial L}{\partial \dot{q}_m}\frac{d\dot{q}_m}{dt}$$

The explicit functional dependence of $L = L(q, \dot{q})$ not t .. so $\frac{dL}{dt} = 0$

$$\cancel{\frac{dL}{dt} = \sum_m \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m} + \frac{\partial L}{\partial \dot{q}_m}\frac{d\dot{q}_m}{dt}}$$

from E-L eq.

$$\frac{dL}{dt} = \sum_m \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m}\dot{q}_m + \frac{\partial L}{\partial \dot{q}_m}\frac{d\dot{q}_m}{dt}$$

$$\frac{dL}{dt} = \sum_m \frac{d}{dt}\left(\dot{q}_m \frac{\partial L}{\partial \dot{q}_m}\right)$$

so,

$$\frac{d}{dt}\underbrace{\left(L - \sum_m \dot{q}_m \frac{\partial L}{\partial \dot{q}_m}\right)}_{\text{a constant} \equiv H} = 0$$

or. $$H = \sum_{m=1}^{N} \overset{\partial L/\partial \dot q}{\overset{\|}{p_m}}\, \dot q_m - L(q, p)$$

$$dH = \dot q_m\, dp_m + \underline{p_m\, d\dot q_m} - \underline{\cancelto{p_m}{\frac{\partial L}{\partial \dot q_m}}\, d\dot q_m} - \frac{\partial L}{\partial q}\, dq_m$$

$$= \dot q_m\, dp_m - \frac{\partial L}{\partial q_m}\, dq_m$$

$$dH = \dot q_m\, dp_m - \dot p_m\, dq_m$$ ~~[illegible]~~

so, H depends on p and q only.

Generally,

$$dH = \frac{\partial H}{\partial p_m}\, dp_m + \frac{\partial H}{\partial q_m}\, dq_m$$

~~giving~~ equating terms

$$\left.\begin{aligned} \frac{\partial H}{\partial p_m} &= \dot q_m = \frac{dq_m}{dt} \\ \frac{\partial H}{\partial q_m} &= -\dot p_m = -\frac{dp_m}{dt} \end{aligned}\right\} \text{Hamilton's Equations}$$

A hint of the future.

$$\frac{dH}{dt} = \frac{\partial H}{\partial p_m}\frac{dp_m}{dt} + \frac{\partial H}{\partial q_m}\frac{dq_m}{dt}$$

$$= \dot q_m \dot p_m - \dot p_m \dot q_m$$ $\rightarrow$ looks like Heisenberg's equations of motion.

Simple example – SHO. $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_m} - \frac{\partial \mathcal{L}}{\partial q_m} = 0$

$T = \frac{1}{2} m\dot{q}^2 \qquad U = \frac{1}{2} kq^2$

$L = T - U = \frac{m\dot{q}^2}{2} - \frac{kq^2}{2}$ where q's may not have anything to do with usual displacements

$\frac{\partial \mathcal{L}}{\partial \dot{q}} = m\dot{q} \qquad \frac{\partial \mathcal{L}}{\partial q} = -kq$

so $$\cancel{\frac{\partial \mathcal{L}}{\partial q}} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q_m} = 0$$

$$\frac{d}{dt} m\dot{q} + kq = 0$$

$$m\ddot{q} + kq = 0$$

for $\omega^2 \equiv k/m$ $\qquad \ddot{q} + \omega^2 q = 0 \rightarrow$ the Newtonian solution.

-- for each degree of freedom, perhaps normal coordinates

$$H = p\dot{q} - L$$
$$= m\dot{q}^2 - \frac{m\dot{q}^2}{2} + \frac{kq^2}{2}$$

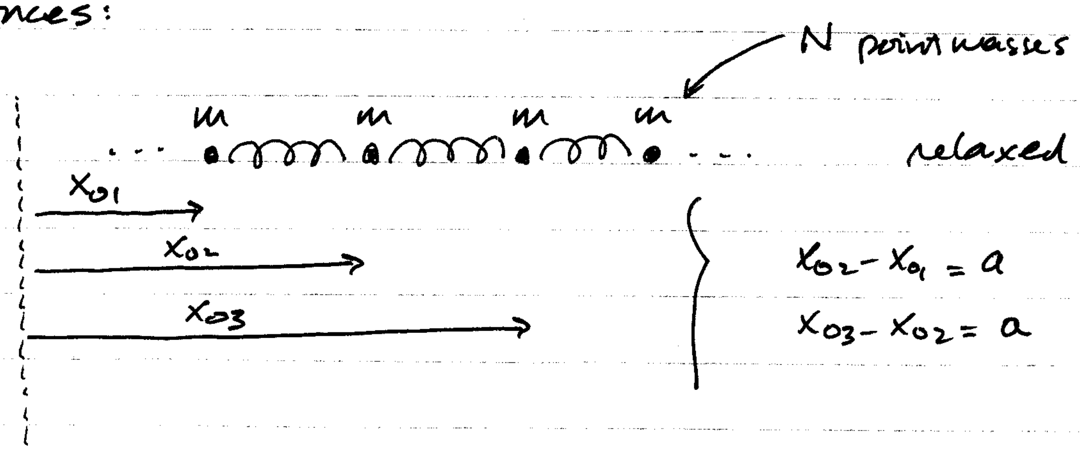
$H = \frac{1}{2}\frac{p^2}{2m} + \frac{kq^2}{2}$ total energy.

Ham Eqs: $\frac{\partial H}{\partial p} = \frac{\partial H}{\partial (m\dot{q})} = \frac{p}{m} = \frac{dq}{dt} \Rightarrow p = m\dot{q}$

$$\frac{\partial H}{\partial q} = kq = -\dot{p} = m\ddot{q} \Rightarrow m\ddot{q} = -kq$$

The passage to continuous systems is relatively straightforward by identifying T & U in a continuous limit.

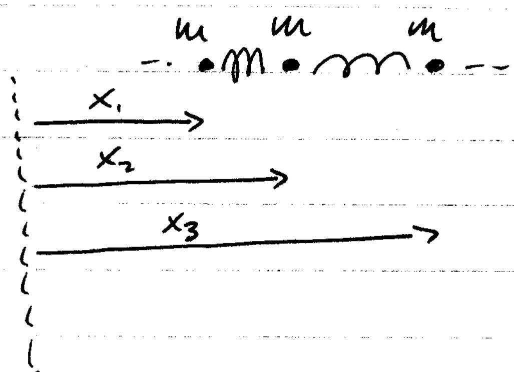
Suppose we have a rod under compressional (or stretching) forces:



$x_{02} - x_{01} = a$

$x_{03} - x_{02} = a$ etc.

An arbitrary unrelaxed situation:



Overall length: $(N-1)a$

mass: Nm

density $\dfrac{Nm}{(N-1)a} = \left(\dfrac{1}{1-1/N}\right)\dfrac{m}{a}$

so as $N \to \infty$, density $\to \dfrac{m}{a}$

Potential energy

$$U = \tfrac{1}{2}k(x_2 - x_1 - a)^2 + \tfrac{1}{2}k(x_3 - x_2 - a) + \ldots + \tfrac{1}{2}k(x_N - x_{N-1} - a)^2$$

$$= \tfrac{1}{2}k \sum_{i=1}^{N-1} (x_{i+1} - x_i - a)^2$$

The appropriate generalized coordinates are relative

displacements
$$\begin{cases} \eta_1 \equiv x_1 - x_{01} & \Rightarrow \quad x_1 = x_{01} + \eta_1 \\ \eta_2 \equiv x_2 - x_{02} & \Rightarrow \quad x_2 = x_{02} + \eta_2 \\ \text{etc.} & \end{cases}$$

so,
$$x_2 - x_1 - a = \underbrace{x_{02} + \eta_2 - x_{01}}_{a} - \eta_1 - a$$

$$= \eta_2 - \eta_1$$

$$U = \tfrac{1}{2}k \sum (\eta_{i+1} - \eta_i)^2$$

so, the generalized coordinate will be the displacement -- or DISTURBANCE (think "excitation")

$$T = \tfrac{1}{2} \sum_{i=1}^{N} m\dot{x}_i^2 \rightarrow \tfrac{1}{2} m \sum_{i=1}^{N} \dot{\eta}_i^2$$

$$L = T - U = \tfrac{1}{2} m \sum_{i=1}^{N} \dot{\eta}_i^2 - \tfrac{1}{2} k \sum_{i=1}^{N-1} (\eta_{i+1} - \eta_i)^2$$

note: $\frac{\partial L}{\partial \dot{\eta}_1} = m\dot{\eta}_1 \qquad \frac{\partial L}{\partial \eta_1} = k(\eta_2 - \eta_1)$

and for the #1 dof:

$$\frac{\partial L}{\partial \eta_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_1} = 0$$

$$k(\eta_2 - \eta_1) - m\ddot{\eta}_1 = 0$$

coupled to #2.
and so on. —

Multiply L by a/a

$$L = \tfrac{1}{2} a \sum \frac{m}{a} \dot{\eta}_i^2 - \tfrac{1}{2} a \sum ka \left(\frac{\eta_{i+1} - \eta_i}{a}\right)^2$$

$$\frac{\partial L}{\partial \dot{\eta}_j} = a\left(\frac{m}{a}\right)\dot{\eta}_j$$

$$\frac{\partial L}{\partial \eta_j} = -\frac{a}{2}\sum_i ka \frac{\partial}{\partial \eta_j}\left(\frac{\eta_{i+1} - \eta_i}{a}\right)^2$$

$$= -a \sum_i ka \left(\frac{\eta_{i+1} - \eta_i}{a^2}\right)\left(\frac{\partial \eta_{i+1}}{\partial \eta_j} - \frac{\partial \eta_i}{\partial \eta_j}\right)$$

$$= -a \sum_i ka \left(\frac{\eta_{i+1} - \eta_i}{a^2}\right)\left(\delta_{j,i+1} - \delta_{ji}\right)$$

$$= -ka^2 \left[\left(\frac{\eta_j - \eta_{j-1}}{a^2}\right) - \left(\frac{\eta_{j+1} - \eta_j}{a^2}\right)\right]$$

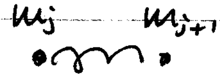
not at ends, $j = 1, N$

so... E-L eq:

$$a\left\{\frac{m}{a}\ddot{\eta}_j + ka\left[\left(\frac{\eta_j - \eta_{j-1}}{a^2}\right) - \left(\frac{\eta_{j+1} - \eta_j}{a^2}\right)\right]\right\} = 0$$

which is a set of coupled equations

Suppose we try to extend one link--



The generalized force required

$$Q_j = -\frac{\partial U}{\partial \eta_j} = k(\eta_{j+1} - \eta_j) = \underline{ka}\left(\frac{\eta_{j+1} - \eta_j}{a}\right)$$

$$Q_j = Y\xi_j$$

elongation per unit length

Young's Modulus $Y = ka$

In the limit in which a gets very small -- the indexes go to differentials and the sums go to integrals.

$$\left.\begin{array}{l} a \to dx \\ m \to dm \end{array}\right\} \quad \frac{m}{a} \to \rho \text{, constant.}$$

$$\eta_j \to \eta(x) \qquad \text{or actually, } \eta(x,t)$$

$$\xi_j \to \frac{\eta(x+a) - \eta(x)}{a} \to \frac{\partial \eta(x)}{\partial x}$$

$$\sum_i \to \int dx$$

so, $$L = \frac{1}{2}\int_{\text{system}} dx \left[\rho\,\dot{\eta}^2(x,t) - Y\left(\frac{\partial \eta(x,t)}{\partial x}\right)^2\right]$$

N.B. x and t are <u>parameters</u>, not any longer degrees of freedom

From Hamilton's Principle,

$$S = \int_{t'}^{t''} dt\, L$$

$$\delta S = 0 = \delta \int dt\, L$$

$$= \delta \int_{t'}^{t''} dt \int_{x'}^{x''} dx\, \frac{1}{2}\left[\rho\, \dot{\eta}^2(x,t) - Y\left(\frac{\partial \eta(x,t)}{\partial x}\right)^2 \right]$$

Define the Lagrange Density, $\mathcal{L}$, such that

$$L = \int dx\, \mathcal{L} \qquad \text{so here} \qquad \mathcal{L} = \tfrac{1}{2}\rho\dot{\eta}^2 - \tfrac{1}{2}Y\left(\frac{\partial \eta}{\partial x}\right)^2$$

and

$$S = \int_{t'}^{t''} dt \int_{x'}^{x''} dx\; \mathcal{L}\Big[\underbrace{\eta(x,t),\ \dot{\eta}(x,t),\ \frac{\partial \eta(x,t)}{\partial x}}_{\text{a functional} \to \text{no } x \text{ or } t.} \Big]$$

Hamilton's Principle says that the motion of this system is according to S is an extremum.

For endpoints fixed:

$$\cancel{\delta S = \int dt \int dx \left\{ \frac{\delta \mathcal{L}}{\delta \eta(x,t)}\, \delta(\eta(x,t)) + \frac{\delta \mathcal{L}}{\delta(\partial \eta/\partial x)}\, \delta(+ \frac{\delta \mathcal{L}}{\delta(}\right.}$$

$$\delta S = \int dt \int dx \left\{ \frac{\delta \mathcal{L}}{\delta \eta(x,t)} \delta \eta(x,t) + \frac{\delta \mathcal{L}}{\delta(\partial\eta/\partial x)} \delta\left(\frac{\partial \eta}{\partial x}\right) \right.$$

$$\left. + \frac{\delta \mathcal{L}}{\delta(\partial\eta/\partial t)} \delta(\partial\eta/\partial t) \right\}$$

do a parts integration twice -- with endpoints fixed:

$$\delta S = \int dt \int dx \left\{ \frac{\delta \mathcal{L}}{\delta \eta} \delta\eta - \frac{\partial}{\partial x}\left[\frac{\delta \mathcal{L}}{\delta(\partial\eta/\partial x)}\right]\delta\eta \right.$$

$$\left. - \frac{\partial}{\partial t}\left[\frac{\delta \mathcal{L}}{\delta(\partial\eta/\partial t)}\right]\delta\eta \right\}$$

$$= \int dt \int dx \left\{ \frac{\delta \mathcal{L}}{\delta \eta} - \frac{\partial}{\partial x}\left[\frac{\delta \mathcal{L}}{\delta(\partial\eta/\partial x)}\right] - \frac{\partial}{\partial t}\left[\frac{\delta \mathcal{L}}{\delta(\partial\eta/\partial t)}\right] \right\}\delta\eta$$

$= 0$ for physical system.

For arbitrary $\delta\eta \Rightarrow$

$$\frac{\delta \mathcal{L}}{\delta \eta(x,t)} - \frac{\partial}{\partial x}\left(\frac{\delta \mathcal{L}}{\delta(\partial\eta/\partial x)}\right) - \frac{\partial}{\partial t}\left(\frac{\delta \mathcal{L}}{\delta(\partial\eta/\partial t)}\right) = 0$$

Euler-Lagrange Equations of motion for continuous systems.

For our system. $\mathcal{L} = \frac{1}{2}\rho\dot{\eta}^2 - \frac{1}{2}Y\left(\partial\eta/\partial x\right)^2$

$$\frac{\delta\mathcal{L}}{\delta\eta} = 0 \qquad \frac{\delta\mathcal{L}}{\delta(\partial\eta/\partial x)} = -Y\left(\partial\eta/\partial x\right)$$

$$\frac{\delta\mathcal{L}}{\delta(\partial\eta/\partial t)} = \rho\frac{\partial\eta}{\partial t}$$

and E-L

$$Y\frac{\partial^2\eta}{\partial x^2} - \rho\frac{\partial^2\eta}{\partial t^2} = 0$$

which is a wave equation for integral disturbances which propagate with velocity $v = \sqrt{Y/\rho}$.

The dynamics is about $\eta(x,t)$, not x, not t.

Further, define Hamiltonian density through

$$H = \int dx\,\mathcal{H}\left(\eta, \dot{\eta}, \frac{\partial\eta}{\partial x}\right)$$

so, $\mathcal{H} = \dot{\eta}\frac{\partial\mathcal{L}}{\partial\dot{\eta}} - \mathcal{L}$

(changing notation to lazy one: $\frac{\delta\mathcal{L}}{\delta\eta} \to \frac{\partial\mathcal{L}}{\partial\eta}$)

Again, have a "momentum"

$$\frac{\partial\mathcal{L}}{\partial\dot{\eta}} = \rho\dot{\eta} \equiv \pi(x,t)$$ canonical momentum which is conjugate to η

obviously, not a mechanical momentum

So, $\mathcal{H} = \dot{\eta}\pi - \mathcal{L}$

$$= \rho\dot{\eta}^2 - \tfrac{1}{2}\rho\dot{\eta}^2 + \tfrac{1}{2}Y\left(\frac{\partial\eta}{\partial x}\right)^2$$

$$= \tfrac{1}{2}\frac{\pi^2}{\rho} - \tfrac{1}{2}Y\left(\partial\eta/\partial x\right)^2$$

Hamilton's Equations:

$$\frac{\partial\mathcal{H}}{\partial\pi} = \dot{\eta}$$

$$\frac{\partial\mathcal{H}}{\partial\pi} = \frac{\pi}{\rho} \implies \pi = \rho\dot{\eta} \quad \checkmark$$

and

$$\frac{\partial\mathcal{H}}{\partial\eta} = -\dot{\pi} = 0$$

Instructive: Longitudinal vibrations in a gas.

A gas w/ equilibrium pressure and density P_o, ρ_o and small fluctuations.

Generalized coordinates: $\vec{\eta}$, a displacement

* START BY FINDING T & U → L → Ham Principle *

$$\mathcal{L} = \mathcal{T} - \mathcal{U} \qquad (\text{densities})$$

$$\mathcal{T} = \frac{\rho_0}{2}\dot{\vec{\eta}}^{\,2} = \frac{\rho_0}{2}\left(\dot{\eta}_1^2 + \dot{\eta}_2^2 + \dot{\eta}_3^2\right)$$ like 3d continuous string.

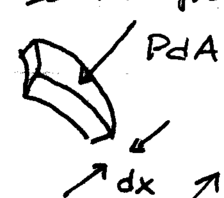
Potential energy requires consideration of the thermodynamic system → work done in expanding gas against pressure...

$$\rho_0 = \frac{M}{V_0} \quad \rightarrow \quad V_0 = \frac{M}{\rho_0}$$

$$U = V_0\,\mathcal{U} \qquad (V_0,\ \text{volume small, so } \mathcal{U} \text{ constant within } V_0)$$

A pressure disturbance causes: $V_0 \rightarrow V_0 + \Delta V$

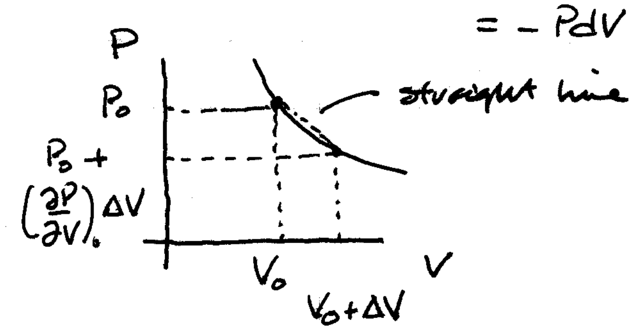
work is done ON the system and so the potential energy INCREASES $= -PdV$



$-PdAdx = dW$
$= -PdV$

So, PE:

$$U = \mathcal{U}V_0 = -\int_{V_0}^{V_0+\Delta V} PdV$$



$$\mathcal{U} = P_0 \Delta V + \frac{1}{2}\left(\frac{\partial P}{\partial V}\right)_0 (\Delta V)^2$$ straight line app.

adiabatic, not isothermal expansion: $PV^\gamma = c$

where $\gamma = c_P/c_V$

so,

$$\left(\frac{\partial P}{\partial V}\right)_0 = -\frac{\gamma P_0}{V_0}$$

and $$\Delta V = -\frac{M}{\rho_0^2}\Delta\rho$$

Now, define $\sigma \equiv \frac{\rho-\rho_0}{\rho_0} = \frac{\Delta\rho}{\rho_0}$ fractional density change

so $\rho = \rho_0(1+\sigma)$

Then, $$\Delta V = -V_0\sigma$$

and

$$\mathcal{U} = \frac{P_0\,\Delta V}{V_0} + \frac{1}{2}\left(-\frac{\gamma P_0}{V_0}\right)\left(\frac{\Delta V}{V_0}\right)^2$$

$$= -P_0\sigma + \tfrac{1}{2}\gamma P_0\sigma^2$$

Need ~~for~~ σ in terms of η...

mass out of V_0 = $\rho_0\int \vec{\eta}\cdot d\vec{A}$

which = mass transport or change in density

$= -\rho_0\int\sigma\, dV = \rho_0\int\vec{\eta}\cdot d\vec{A}$

The divergence theorem: $-\int\sigma\, dV = \int\vec{\nabla}\cdot\vec{\eta}\, dV$

So, $$\sigma = -\vec{\nabla}\cdot\vec{\eta}$$

and $\sigma = -\vec{\nabla}\cdot\vec{\eta}$ gives

$$\mathcal{U} = -P_0\,\vec{\nabla}\cdot\vec{\eta} + \frac{\gamma P_0}{2}\,(\vec{\nabla}\cdot\vec{\eta})^2$$

so: $\mathcal{L} = \frac{1}{2}\left(\rho_0\dot{\vec{\eta}}^{\,2} + 2P_0\,\vec{\nabla}\cdot\vec{\eta} - \gamma P_0\,(\vec{\nabla}\cdot\vec{\eta})^2\right) = KE - PE$

The E-L eq: $\dfrac{\partial\mathcal{L}}{\partial\dot{\eta}_i} = \rho_0\dot{\eta}_i$

each term... $\dfrac{\partial(\vec{\nabla}\cdot\vec{\eta})}{\partial(\partial\eta_i/\partial x_k)} = \delta_{ik}$

$$\frac{\partial(\vec{\nabla}\cdot\vec{\eta})^2}{\partial(\partial\eta_i/\partial x_k)} = 2\,(\vec{\nabla}\cdot\vec{\eta})\,\delta_{ik}$$

and in components:

$$\rho_0\frac{\partial^2\eta_i}{\partial t^2} - \gamma P_0\frac{\partial(\vec{\nabla}\cdot\vec{\eta})}{\partial x_i} = 0$$

As vector equation

$$\rho_0\frac{\partial^2\vec{\eta}}{\partial t^2} - \gamma P_0\,\vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{\eta}) = 0$$

not recognizable... take divergence, remembering $\sigma = -\vec{\nabla}\cdot\vec{\eta}$

$$\nabla^2\sigma - \frac{\rho_0}{\gamma P_0}\frac{\partial^2\sigma}{\partial t^2} = 0$$

a 3d wave equation in the fractional density change w/

$$v = \sqrt{\frac{\gamma P_0}{\rho_0}} = v_{SOUND}$$

So what? Started from the KE and PE of parts of a material substance -- the gas.

ANOTHER WAY TO LOOK AT IT!

Get rid of the gas and leave behind the scalar quantity which is the value of σ at every x, y, z, t

a scalar field -- no mechanical system.

Turn the process around! assume the equations of motion — the well known wave equation — and FIND THE $\mathcal{L}$ THAT GIVES IT

Here's one: $$\mathcal{L} = \frac{1}{2}\left(\frac{\rho_0}{\gamma P_0}\dot{\sigma}^2 - (\vec{\nabla}\sigma)^2 \right)$$

It's not quite the same as the first approach, but it gives the right equation of motion.

NO Kinetic energy
NO Potential energy

This is where we start in Field Theory.