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$$\hat{\mathcal{L}}(x) = \frac{1}{2}\frac{\partial\hat\phi}{\partial x^\mu}\frac{\partial\hat\phi}{\partial x_\mu} - \frac{1}{2}m^2\hat\phi^2$$

and we get the K.G. equation $\frac{\partial^2\varphi}{\partial x_\mu \partial x^\mu} + m^2\varphi = 0$

and Hamiltonian

$$\hat{\mathcal{H}} = \hat\pi\dot{\hat\phi} - \hat{\mathcal{L}}$$
$$= \frac{1}{2}\hat\pi^2 + \frac{1}{2}\vec\nabla\hat\varphi\cdot\vec\nabla\hat\varphi + \frac{1}{2}m\hat\varphi^2 \quad \rightarrow$$ operators all give + definite eigenvalues $\Rightarrow$ no negative E

-- like relativistic Harmonic Oscillator $\Pi^2 + \omega^2 X^2$

w/ $\omega^2 = k^2 + m^2$

so, we quantize by demanding

$$[\pi_i(\vec x,t), \varphi_j(\vec x',t)] = -i\,\delta_{ij}\,\delta(\vec x - \vec x')$$

$$[\pi_i(\vec x,t), \pi_j(\vec x,t)] = [\varphi_i(\vec x,t), \varphi_j(\vec x',t)] = 0$$

same time

calculate $[\hat H, \hat\phi(\vec x',t)]$

drop $\hat{}$ $\rightarrow$ operators now

$$= \int d^3x\, [\mathcal{H}, \phi(x',t)]$$
$$= \int d^3x \left[\left(\frac{1}{2}\pi(\vec x,t)\pi(\vec x,t) + \frac{1}{2}\vec\nabla\varphi\cdot\vec\nabla\varphi + \frac{1}{2}m^2\phi(\vec x,t)\phi(\vec x,t)\right), \phi(\vec x',t)\right]$$

suppress spacetime dependence: $\phi' \equiv \phi(\vec x',t)$ etc

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$$= \frac{1}{2}\int d^3x \; \Big(\pi\pi\phi' + \underline{\vec{\nabla}\phi\cdot\vec{\nabla}\phi\,\phi'} + \underline{m^2\phi\phi\phi'}$$
$$- \phi'\pi\pi - \underline{\phi'\vec{\nabla}\phi\cdot\vec{\nabla}\phi} - \underline{m^2\phi'\phi\phi} \Big)$$

since $[\phi, \phi'] = 0$

$$= \frac{1}{2}\int d^3x \; \left(\pi\underline{\pi\phi'} - \underline{\phi'\pi}\pi \right)$$

note

$$[\pi, \phi'] = \pi\phi' - \phi'\pi = -i\,\delta^3(\vec{x}-\vec{x}')$$
$$\pi\phi' = \phi'\pi - i\delta^3(\vec{x}-\vec{x}')$$

$$H = \frac{1}{2}\int d^3x \; \left(\underline{\pi\phi'\pi} - i\pi\,\delta^3(\vec{x}-\vec{x}') - \underline{\pi\phi'\pi} - \pi\, i\,\delta^3(\vec{x}-\vec{x}') \right)$$

$$= \frac{1}{2}\int d^3x \; i\, 2\pi(\vec{x},t)\,\delta^3(\vec{x}-\vec{x}')\,(-i)$$

$$= -i\pi(\vec{x}',t) = -i\dot{\phi}(\vec{x}',t)$$

so $\quad [H, \phi(\vec{x},t)] = -i\dfrac{\partial\phi}{\partial t}(\vec{x}',t)$

$\Rightarrow$ field operators satisfy Heisenberg's equations of motion – which characterized the time dependence.

Normalization: many conventions to deal with and the need to pass from a box normalization (Σ) to continuous normalization ($\int$).

Remember that we naturally found a normalization for $\vec{A} \propto \frac{1}{\sqrt{2\omega V}} (\quad)$. This is part of the story, but not enough. By allowing the "box" to become unbounded, in essence, our Fourier series representation with standing wave solutions $\rightarrow$ Fourier integral representation for traveling wave solutions. So this would suggest

$$\phi(\vec{x}, t) \propto \int_{-\infty}^{\infty} \frac{d^3k}{2\pi \cdot \text{stuff}} \left(a(k) e^{-ik\cdot x} + hc \right)$$

The most modern way to assign "stuff" is to use

$$\boxed{\int \frac{d^3k}{(2\pi)^3 2\omega_k}} \qquad \text{because} \qquad \int \frac{d^3k}{2\omega_k} \text{ is}$$

Lorentz invariant. which can be seen by noting

$$d^3k\, dk_0$$

$$\frac{d^3k}{2\omega_k} = d^4k\, \delta(k^2 - m^2)\, \theta(k_0) \qquad \text{positive energies only}$$

remember that, under an integral,

$$\delta[f(x)] = \sum_i \frac{\delta(x - x_i)}{|df/dx|_{x=x_i}} \qquad \text{where } x_i \text{ are the simple zeros of } f(x_i) = 0$$

here, $f = k^2 - m^2 = f(k_0)$

$= k_0^2 - \vec{k}^2 - m^2$ find $f = 0$:

$k_0 = \pm\sqrt{\vec{k}^2 + m^2}$

$= \pm\, \omega_k$ zeros.

so $df/dk_0 = 2k_0$

$\left| df/dk_0 \right|_{k_0 = \omega_k} = 2\omega_k$

and $\delta(k^2 - m^2) = \dfrac{\sum_{i=\pm} \delta(k_0 - \omega_{k_i})}{2\omega_k}$

$= \dfrac{\delta(k_0 + \omega_k) + \delta(k_0 - \omega_k)}{2\omega_k}$

$$(2\pi)\frac{d^4k}{(2\pi)^4}\,\delta(k^2 - m^2)\,\theta(k_0) = \frac{d^4k}{2\omega_k(2\pi)^3}\left[\delta(k_0 + \omega_k) + \delta(k_0 - \omega_k)\right]\theta(k_0)$$

$$= \frac{d^3\vec{k}\, dk_0}{(2\pi)^3 2\omega_k}\,\delta(k_0 - \omega_k)$$

$$= \frac{d^3k}{(2\pi)^3 2\omega_k}\Bigg|_{k_0 = \omega_k \text{ as a condition}}$$

$\omega_k = \sqrt{\vec{k}^2 + m^2}$

So, we have

$$\phi(x) = \int \frac{d^4k}{(2\pi)^3}\,\delta(k^2 - m^2)\, f_k(x)$$

where $f_k(x) = A(k)\,e^{-ik\cdot x}$ a general solution to K.G. equation

$$\phi(x) = \int \frac{d^4k}{(2\pi)^3}\,\delta(k^2 - m^2) A(k)\, e^{-ik\cdot x}$$

from above

$$\phi(x) = \int \frac{d^3k\,dk_0}{(2\pi)^3 2\omega_k} A(k_0,\vec{k})\left[\delta(k_0+\omega_k) + \delta(k_0-\omega_k)\right] \underbrace{e^{-ik\cdot x}}_{= e^{-i\omega t + i\vec{k}\cdot\vec{x}}}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[A(-\omega_k,\vec{k})\,e^{i(\omega_k t+\vec{k}\cdot\vec{x})} + A(\omega_k,\vec{k})\,e^{-i(\omega_k t-\vec{k}\cdot\vec{x})}\right]$$

write

$$A(-\omega_k,\vec{k})\,e^{i\vec{k}\cdot\vec{x}} = A(-\omega_k,-\vec{k})\,e^{-i\vec{k}\cdot\vec{x}}$$

since $k_0^2 = \vec{k}^2 + m^2$ for either sign of k_0,

$$A(\omega_k,\vec{k}) = A(\vec{k}) \equiv a(\vec{k})$$
$$A(-\omega_k,-\vec{k}) = A(-\vec{k}) \equiv a(-\vec{k})$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[a(-\vec{k})\,e^{+ik\cdot x} + a(\vec{k})\,e^{-ik\cdot x}\right]$$

also $\quad \phi^+(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[a^+(-\vec{k})\,e^{-ik\cdot x} + a^+(\vec{k})\,e^{ik\cdot x}\right]$

for a <u>real</u> field, $\quad \phi = \phi^+$, so $\quad a^+(-\vec{k}) = a(\vec{k})$

$$a^+(\vec{k}) = a(-\vec{k})$$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[\underbrace{a(\vec{k})\,e^{-ik\cdot x}}_{+\text{ energy}} + \underbrace{a^+(\vec{k})\,e^{ik\cdot x}}_{-\text{ energy}}\right]$$

another way to look at it ...

start with commutation relations – quantization condition –

showed : $[\hat{H}, \hat{\varphi}(\vec{x}',t)] = -i\dot{\hat{\varphi}}(x',t)$

likewise: $[\hat{H}, \hat{\pi}(\vec{x}',t)] = -i\dot{\hat{\pi}}(x',t)$

$= -i(\vec{\nabla}'^2 - m^2)\hat{\varphi}(x',t)$

So,

$$\star \quad (\vec{\nabla}^2 - m^2)\hat{\varphi} = \dot{\hat{\pi}} = \ddot{\hat{\varphi}}$$ Operators satisfy K.G.E.

Presume a basis of expansion

$$\xi_k = A(k)e^{+i\vec{k}\cdot\vec{x}}$$

so $$\hat{\varphi}(\vec{x},t) = \int d^3k\, A(k)e^{+ik\cdot x}\,\hat{a}_k(t)$$

substitution into $\star$

$$\ddot{\hat{a}}_k(t) = -(\vec{k}^2 + m^2)\,\hat{a}_k(t)$$

for which a general solution is

$$\hat{a}_k(t) = \hat{a}(\vec{k})e^{-i\omega_k t} + \hat{b}(k)e^{i\omega_k t}$$

$$\hat{\varphi}(x) = \int d^3k\, A(k) \left[\hat{a}(\vec{k}) e^{-i\omega_k t + i\vec{k}\cdot\vec{x}} + \hat{b}(\vec{k}) e^{i\omega_k t + i\vec{k}\cdot\vec{x}} \right]$$

$$\hat{\varphi}^{+} = \int d^3k\, A(k) \left[\hat{a}^{+}(\vec{k}) e^{i\omega_k t + i\vec{k}\cdot\vec{x}} + \hat{b}^{+}(\vec{k}) e^{-i\omega_k t + i\vec{k}\cdot\vec{x}} \right]$$

For a real field $\varphi = \varphi^{+}$
and

$$\hat{b}(-\vec{k}) = \hat{a}^{+}(\vec{k})$$

so,

$$\hat{\varphi}(x) = \int d^3k\, A(k) \left[\hat{a}(\vec{k}) e^{-ik\cdot x} + \hat{a}^{+}(\vec{k}) e^{ik\cdot x} \right]$$

The normalization comes from the realization that

$$\frac{d^3k}{2\omega_k} = d^4k\, \delta(k^2 - m^2)\, \theta(k_0)$$

showed this was so —

and we have for a real field

$$\hat{\varphi}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[a(\vec{k}) e^{-ik\cdot x} + a^{+}(\vec{k}) e^{ik\cdot x} \right]$$

We can figure out the algebra of the a's. It's standard to define the "positive (and negative) frequency solutions" to properly define the a's:

$$f_k(x) \equiv \frac{1}{\left[(2\pi)^3 2\omega_k\right]^{1/2}} e^{-ik\cdot x}$$

+ frequency component

− freq comp.

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left[a(\vec{k}) f_k(x) + a^\dagger(\vec{k}) f_k^*(x) \right]$$

$$\equiv \phi^{(+)}(x) + \phi^{(-)}(x)$$

form, $\int f_{k'}^*(x)\,\phi(x)\,d^3x = \int d^3x \frac{1}{\sqrt{}} \int \frac{d^3k}{(2\pi)^3 2\omega_k}$

$$\left[a(\vec{k}) \underbrace{e^{ik'\cdot x} e^{-ik\cdot x}}_{} + a^\dagger(\vec{k}) e^{ik'\cdot x} e^{ik\cdot x} \right]$$

$$e^{ik_0' x^0 - i\vec{k}'\cdot\vec{x}}\, e^{-ik^0 x^0 + \vec{k}\cdot\vec{x}} + \underbrace{}_{\text{etc.}}$$

so (define $dK \equiv \dfrac{d^3k}{(2\pi)^3 2\omega_k}$)

$$= \int d^3x \frac{1}{\sqrt{}} \int dK \Big[a(\vec{k})\, e^{i(k^{0\prime}-k^0)x^0 - i(\vec{k}'-\vec{k})\cdot\vec{x}}$$

$$+ a^\dagger(\vec{k})\, e^{i(k^{0\prime}+k^0)x^0 - i(\vec{k}'+\vec{k})\cdot\vec{x}} \Big]$$

integrate $\int d^3\vec{x}$

$$= \frac{1}{\sqrt{}} \int dK \Big[a(\vec{k})\, e^{i(k_0'-k_0)t} (2\pi)^3 \delta(\vec{k}'-\vec{k})$$

$$+ a^\dagger(\vec{k})\, e^{i(k_0'+k_0)t} (2\pi)^3 \delta(\vec{k}'+\vec{k}) \Big]$$

$$= \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} \frac{1}{(2\pi)^3 2\omega_k} (2\pi)^3 \Big[a(\vec{k}')\, e^{i(k_0'-k_0)t} + a^\dagger(-\vec{k})\, e^{i(k_0'+k_0)t} \Big]$$

note — δ integration forces $|\vec{k}| = |\vec{k}'|$

and $m^2 = \vec{k}'^2 + k^{0\prime 2} = \vec{k}^2 + k^{0\prime 2}$

common mass $\Rightarrow k^{0\prime} = k^0 = \omega_k$.

$$= \frac{(2\pi)^3}{[\quad]^{3/2}} \Big[a(\vec{k}') + a^\dagger(-\vec{k})\, e^{2i\omega_k t} \Big]$$

$$\int f^*_{k'}(x)\phi(x)\, d^3x = \frac{1}{\sqrt{}} \frac{1}{2\omega_k} \Big[a(\vec{k}') + a^\dagger(-\vec{k})\, e^{2i\omega_k t} \Big]$$

Similarly, calculate

$$\int f^*_{k'}(x)\, \dot{\phi}(x)\, d^3x = \frac{-i}{2} \frac{1^{1/2}}{\sqrt{}} \Big[a(\vec{k}') - a^\dagger(-\vec{k})\, e^{2i\omega t} \Big]$$

now, isolate the a's

$$\int f^*_{k'}(x) \big[\omega_k \phi(x) + i \dot{\phi}(x) \big] d^3x = \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} \, a(\vec{k}')$$

so,

$$a(\vec{k}') = \int f^*_{k'}(x)\left[\omega_{k'}\phi(x) + i\dot{\phi}(x)\right]\sqrt{(2\pi)^3 2\omega_{k'}}\; d^3x$$

$$= i\int d^3x \sqrt{} \left[-\frac{\partial f^*_{k'}}{\partial t}\phi(x) + f^*_{k'}(x)\frac{\partial\phi(x)}{\partial t}\right]$$

$$\equiv i\int d^3x \sqrt{} \left(f^*_{k'}(x)\overleftrightarrow{\partial_0}\phi(x)\right)$$

where $\quad A\overleftrightarrow{\partial_0}B \equiv A\partial_0 B - (\partial_0 A)B$

Likewise, $\quad a^\dagger(\vec{k}) = -i\int d^3x \sqrt{}\; f_k(x)\overleftrightarrow{\partial_0}\phi(x)$

So, now work out the algebra of the a's,

$$[a(k), a^\dagger(k')] = +\int d^3x \int d^3x'\,(2\pi)^3\sqrt{2\omega_k 2\omega_{k'}}\left[f^*_k(x)\overleftrightarrow{\partial_0}\phi(x), f_{k'}(x')\overleftrightarrow{\partial_0}\phi(x')\right]$$

look at commutator,

$$[\;,\;] = \left[f^*_k(x)\frac{\partial\phi(x)}{\partial t} - \frac{\partial f^*_k(x)}{\partial t}\phi(x), \quad \text{etc.} \right]$$

use shorthand $\quad f_{k'}(x') \equiv f'$

$\qquad\qquad\qquad \phi(x') \equiv \phi'$

$$= [f^*\dot{\phi} - \dot{f}^*\phi,\; f'\dot{\phi}' - \dot{f}'\phi']$$

$$= f^*\dot{\phi} f'\dot{\phi}' - f^*\dot{\phi}\dot{f}'\phi' - \dot{f}^*\phi f'\dot{\phi}' + \dot{f}^*\phi\dot{f}'\phi'$$
$$- f'\dot{\phi}' f^*\dot{\phi} + \dot{f}'\phi' f^*\dot{\phi} + f'\dot{\phi}'\dot{f}^* - \dot{f}'\phi'\dot{f}^*\phi$$

f's are functions, $\quad \hat{\dot{\phi}} = \hat{\pi}$

$$= f^*f'\pi\pi' - f^*\dot{f}'\,\underline{\pi\phi'} - \dot{f}^*f'\,\underline{\underline{\phi\pi'}} + \dot{f}^*\dot{f}'\phi\phi'$$
$$- f'f^*\pi'\pi + \dot{f}'f^*\,\underline{\phi'\pi} + f'\dot{f}^*\,\underline{\underline{\pi'\phi}} - \dot{f}'\dot{f}^*\phi'\phi$$
$$= -\dot{f}'f^*\,\underline{[\pi,\phi']} + f'\dot{f}^*\,\underline{\underline{[\pi',\phi]}} + \underline{\underline{0}}$$

$$= i\dot{f}'f^*\delta(\vec{x}-\vec{x}') - i f'\dot{f}^*\delta(\vec{x}-\vec{x}')$$

so,

$$[a(\vec{k}), a^\dagger(\vec{k}')] = +i\int d^3x\int d^3x'\,(2\pi)^3\sqrt{2\omega_k 2\omega_{k'}}$$

$$\left[\dot{f}'f^*\delta(\vec{x}-\vec{x}') - f'\dot{f}^*\delta(\vec{x}-\vec{x}')\right]$$

$$= i\int d^3x\,(2\pi)^3\sqrt{2\omega_k 2\omega_{k'}}\left[\dot{f}'f^* - f'\dot{f}^*\right]$$

$$= i\int d^3x\,(2\pi)^3\sqrt{2\omega_k 2\omega_{k'}}\;\times$$

$$\left[\left(\frac{-i\omega_{k'}e^{-ik'\cdot x}}{\sqrt{'}}\right)\left(\frac{e^{ik\cdot x}}{\sqrt{}}\right)\right.$$

$$\left.-\left(\frac{e^{-ik\cdot x}}{\sqrt{'}}\right)\left(\frac{i\omega_k e^{ik\cdot x}}{\sqrt{}}\right)\right]$$

$$= i\left[-i\underline{\omega_{k'}}\,e^{-i(k^{0\prime}-k^0)t}(2\pi)^3\delta^3(\vec{k}'-\vec{k}) - i\underline{\omega_k}\,e^{-i(k^{0\prime}-k^0)t}(2\pi)^3\delta^3(\vec{k}-\vec{k}')\right]$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = 2\omega_k(2\pi)^3\delta^3(\vec{k}'-\vec{k})$$

normalization comes from our covariant $A(\vec{k})$

(by working closed and realizing that $\delta(\vec{k}'-\vec{k})$ forces $\vec{k}=\vec{k}' \Rightarrow k^0 = k^{0\prime}$)

Likewise $\quad [a(\vec{k}), a(\vec{k}')] = [a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0$

The Hamiltonian ...

$$\mathcal{H} = \tfrac{1}{2}\left[\pi^2 + (\vec{\nabla}\varphi)^2 + m^2\varphi^2\right]$$

so $\quad H = \int d^3x\, \mathcal{H} = \tfrac{1}{2}\int d^3x \left[\qquad \right]$

$$\phi(x) = \int d\hat{K}\left[a(k)e^{-ik\cdot x} + a^\dagger(k)e^{ik\cdot x}\right]$$

$$\pi(x) = \dot{\phi}(x) = \int d\hat{K}(-i\omega_k)\left[a(k)e^{-ik\cdot x} - a^\dagger(k)e^{ik\cdot x}\right]$$

$$\vec{\nabla}\phi = \int d\hat{K}\,(i\vec{k})\left[a(k)e^{-ik\cdot x} - a^\dagger(k)e^{ik\cdot x}\right]$$

so, $\quad H = \tfrac{1}{2}\int d^3x \int d\hat{K} \int d\hat{K}'$

$$\times\Big\{(-i\omega_k)(-i\omega_{k'})\left[a e^{-ik\cdot x} - a^\dagger e^{ik\cdot x}\right]\left[a' e^{-ik'\cdot x} - a^{\dagger\prime} e^{ik'\cdot x}\right]$$

$$+ (i\vec{k})\cdot(i\vec{k}')\left[\quad\right]\left[\quad\right]$$

$$+ m^2\left[a e^{-ik\cdot x} + a^\dagger e^{ik\cdot x}\right]\left[a' e^{-ik'\cdot x} + a^{\dagger\prime} e^{ik\cdot x}\right]\Big\}$$

we know that H is a constant of the motion $\Rightarrow \dot{H} = 0$
so we can choose a convenient time to do this at --
$t = 0$

$$H = \frac{1}{2}\int d^3x \int dK \int dK'$$

$$\times \Big\{ (-i\omega_k)(-i\omega_{k'})\left[a e^{i\vec{k}\cdot\vec{x}} - a^\dagger e^{-i\vec{k}\cdot\vec{x}}\right]\left[a' e^{i\vec{k}'\cdot\vec{x}} - a^{\dagger\prime} e^{-i\vec{k}'\cdot\vec{x}}\right]$$

$$+ \quad (i\vec{k})\cdot(i\vec{k}')\left[\quad\right]\left[\quad\right]$$

$$+ \quad m^2\left[a e^{i\vec{k}\cdot\vec{x}} + a^\dagger e^{-i\vec{k}\cdot\vec{x}}\right]\left[a' e^{i\vec{k}'\cdot\vec{x}} + a^{\dagger\prime} e^{-i\vec{k}'\cdot\vec{x}}\right]\Big\}$$

$$[\ ][\ ] = aa' e^{i(\vec{k}+\vec{k}')\cdot\vec{x}} - aa^{\dagger\prime} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}$$

$$- a^\dagger a' e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} + a^\dagger a^{\dagger\prime} e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}}$$

again, recognize δ's coming out

$$\int d^3x\, e^{\pm i(\vec{k}+\vec{k}')\cdot\vec{x}} = (2\pi)^3\delta(\vec{k}+\vec{k}') \Rightarrow \vec{k} = -\vec{k}'$$

$$\int d^3x\, e^{\pm i(\vec{k}-\vec{k}')\cdot\vec{x}} = (2\pi)^3\delta(\vec{k}-\vec{k}') \Rightarrow \vec{k} = \vec{k}'$$

so, integrating over the $\int d^3x$ gives, after some work...

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[\frac{1}{2}\omega_k\left(a^\dagger(\vec{k})a(\vec{k}) + a(\vec{k})a^\dagger(\vec{k})\right)\right]$$

which looks like our EM oscillator Hamiltonian, so we're free to make the same interpretation of the operators $a(\vec{k})$ and $a^\dagger(\vec{k})$

... we create and annihilate massive spin ϕ quanta one at a time starting from the vacuum ~~having~~ ~~moment~~

We need to rearrange our energy scale again -- just a little. —

$$[a(k), a^{+}(k)] = (2\pi)^3 2\omega_k \, \delta(0)$$

$$H = \int dK \left\{ \tfrac{1}{2}\omega_k \left[a^{+}a + a^{+}a + (2\pi)^3 2\omega_k \, \delta(0) \right] \right.$$

$$= \int dK \, \omega_k \, a^{+}(\vec{k}) a(\vec{k}) + \underbrace{\int_{-\infty}^{\infty} d^3k \, \delta(0)}_{\text{an infinite offset} \, \equiv \, H_\infty}$$

subtract H_∞ -- from now on

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \, \omega_k \, a^{+}(\vec{k}) a(\vec{k})$$

We can insure this adjustment of the energy scale through an overt ordering of the product of operators... to wit:

our new choice of energy scale ensures $\langle 0|H|0\rangle = 0$

since this is $\langle 0|a^{+}a|0\rangle = 0$

to insure this, we used the commutator, then shifted the energy scale by H_∞

the same thing could be insured by <u>changing the order of all products of a and a+</u> so that always:

a's are to the RIGHT of a^+'s

This is called <u>Normal</u> <u>Ordering</u> --

Remember $\phi = \phi^{(+)} + \phi^{(-)}$

+ freq
a

− freq
a^+

so arranging fn + to the RIGHT of −

eg. $\phi(x)\phi(y) = (\phi^{(+)}(x) + \phi^{(-)}(x))(\phi^{(+)}(y) + \phi^{(-)}(y))$

$$\equiv (x^+ + x^-)(y^+ + y^-)$$

$$= \underset{a\,a}{x^+y^+} + \underset{a\,a^+}{x^+y^-} + \underset{a^+a}{x^-y^+} + \underset{a^+a^+}{x^-y^-}$$

like $aa^+ \Rightarrow \langle 0|x^+y^-|0\rangle \neq 0$

normal ordered products are designated

$$:\phi(x)\,\phi(y): \;=\; x^+y^+ + \underline{\underline{y^-x^+}} + x^-y^+ + x^-y^-$$

$$= \phi^{(+)}(x)\,\phi^{(+)}(y) + \phi^{(-)}(y)\,\phi^{(+)}(x) + \phi^{(-)}(x)\,\phi^{(+)}(y) + \phi^{(-)}(x)\,\phi^{(-)}(y)$$

so by shifting the energy scale as we did we've done

$$H \rightarrow \;:H:$$ — this will happen everywhere

One particle normalization:

let $\quad |\vec{k}\rangle = a^+(\vec{k})\,|0\rangle$

then $\quad \langle \vec{k}|\vec{k}'\rangle = \langle 0|\,a(\vec{k})\,a^+(\vec{k}')\,|0\rangle$

$$= \langle 0|\,[a(\vec{k}), a^+(\vec{k}')]\,|0\rangle + \underbrace{\langle 0|\,a^+(\vec{k}')\,a(\vec{k})\,|0\rangle}_{=0}$$

$$= (2\pi)^3\, 2\omega_k\, \delta(\vec{k}-\vec{k}')\underbrace{\langle 0|0\rangle}_{=1}$$

$$\langle \vec{k}|\vec{k}'\rangle = (2\pi)^3\, 2\omega_k\, \delta(\vec{k}-\vec{k}')$$

called covariant normalization