We can make touch with regular quantum mechanics --

Consider the object

$$\langle 0|\phi(x)|\vec{k}'\rangle = \int dK \langle 0| a(\vec{k})e^{-ik\cdot x} + a^{\dagger}(\vec{k})e^{ik\cdot x}|\vec{k}'\rangle$$

$$= \int dK \langle 0|(a(k)e^{-ik\cdot x} + a^{\dagger}(k)e^{ik\cdot x})a^{\dagger}(\vec{k}')|0\rangle$$

$$= \int dK \langle 0| a(k)a^{\dagger}(k')e^{-ik\cdot x} + \underbrace{a^{\dagger}(k)a^{\dagger}(k')e^{ik\cdot x}|0\rangle}_{=0}$$

$$= \int dK \langle 0|a(k)a^{\dagger}(k')|0\rangle e^{-ik\cdot x}$$

$$= \int dK \langle\vec{k}|\vec{k}'\rangle e^{-ik\cdot x}$$

$$= \int dK (2\pi)^3 2\omega_k \delta(\vec{k}-\vec{k}')e^{-ik\cdot x}$$

$$= \int d^3k\, \delta(\vec{k}-\vec{k}')e^{-ik\cdot x}$$

$$\langle 0|\phi(x)|\vec{k}'\rangle = e^{-i\omega_{k'}t+i\vec{k}'\cdot\vec{x}}$$

a plane wave in +x direction in time for a particle with

$$\omega_{k'}^2 = \vec{k}'^2 + m^2$$

- a de Broglie wave - or regular wave function - of a free particle

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$$\text{So } \psi_{\vec{k}}(x) = \langle 0|\hat{\phi}(x)|0,0,\ldots\vec{k}'\ldots 0\ldots\rangle$$ change of rep from Fock to coord.

normalization aside

$$\langle k | k' \rangle = (2\pi)^3 2\omega_k \, \delta(\vec{k} - \vec{k}')$$

the single particle wavefunction -

$$\psi(x) = \langle 0 | \varphi(x) | k \rangle = e^{-ik\cdot x}$$

Calculate the resultant single particle wave function normalization

$$\int \psi^+(x)\psi(x)\,dx \equiv N \qquad \text{wavefunction normalization}$$

$$= \int d^3x \, \langle k | \varphi^+(x) | 0 \rangle \langle 0 | \varphi(x) | k \rangle$$
$$= \int d^3x \int dP \int dP' \, \langle k | p \rangle \langle p' | k \rangle \, e^{ip\cdot x} e^{-ip'\cdot x}$$

$$= \int dP \int dP' \, \langle k | p \rangle \langle p' | k \rangle \, \delta(\vec{p} - \vec{p}')$$
$$= \int dP \, \underbrace{\langle k | p \rangle \langle p | k \rangle}$$

$|p\rangle$ are a complete set, so

$$\int \psi^+ \psi \, d^3x = \langle k | k \rangle = (2\pi)^3 2\omega_k \, \delta(0)$$

what is $\delta(0)$?

$$\delta(k) \equiv \lim_{L\to\infty} \iiint_{-L/2}^{L/2} dx\,dy\,dz \, \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^3}$$

$$\Rightarrow \quad \delta(0) = \lim_{L\to\infty} \iiint \frac{d^3x}{(2\pi)^3} = \lim_{L\to\infty} \frac{L^3}{(2\pi)^3}$$

call $\quad \lim_{L\to\infty} L^3 \equiv V$

so $\quad \langle k|k\rangle = (2\pi)^3\, 2\omega_k \dfrac{V}{(2\pi)^3} \quad \Rightarrow$

$$\frac{\langle k|k\rangle}{V} = 2\omega_k = \frac{N}{V} \qquad \text{\# particles per unit volume.}$$

Remember density of states?

The number of states per energy interval usually

$$\rho(E)\,dE = \frac{V\,d^3k}{(2\pi)^3} \qquad \text{for 1 particle / unit volume}$$

we have $2E$ particles / unit volume, so the phase space that we use is

$$\rho(E)\,dE = \frac{V\,d^3k}{(2\pi)^3\, 2E}$$

which makes $\int \psi^\dagger \psi \, d^3x$ Lorentz invariant.

For a complex scalar field, remember we had,

$$\phi = \sqrt{1/2}\,(\varphi_1 + i\phi_2)$$

$$\mathcal{L} = \left(\frac{\partial\phi}{\partial x^\mu}\right)^*\left(\frac{\partial\varphi}{\partial x_\mu}\right) - m^2\varphi^*\varphi$$

now:

$$\hat{\phi}(x) = \int dK\,\left[a(\vec{k})e^{-ik\cdot x} + b^\dagger(\vec{k})e^{ik\cdot x}\right]$$

$$\hat{\varphi}^\dagger(x) = \int dK\,\left[b(\vec{k})e^{ik\cdot x} + a^\dagger(\vec{k})e^{-ik\cdot x}\right]$$

These operators have the separate commutation rules,

$$[a(k), a^\dagger(k)] = (2\pi)^3\, 2\omega_k\, \delta(\vec{k}-\vec{k}')$$

$$[b(k), b^\dagger(k)] = (2\pi)^3\, 2\omega_k\, \delta(\vec{k}-\vec{k}')$$

The Hamiltonian is constructed normally.

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L}$$

where $\quad \pi(x) = \dot{\phi}^\dagger = i\int dK\, i\omega_k\left[b(k)e^{-ik\cdot x} - a^\dagger(k)e^{ik\cdot x}\right]$

By now we know how to write it down —

$$:H: = \int dK\, \omega_k \left[a^\dagger(\vec{k})a(\vec{k}) + b^\dagger(\vec{k})\, b(\vec{k}) \right]$$

Let's look at the Noether current and the symmetry.

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial \varphi_i / \partial x^\mu)} \delta\varphi$$

For classical 2d complex fields and $U(1)$ transformations $\delta\varphi = -i\alpha\varphi$

$$\mathcal{L} = \partial^\mu \varphi^* \partial_\mu \varphi - m^2 \varphi^* \varphi$$

$$\rightarrow \qquad J^\mu(x) = i\,(\varphi^* \partial^\mu \varphi - \varphi\, \partial^\mu \varphi^*)$$

So in the quantized theory we would have

$$:J^\mu(x): \;=\; i :\hat{\varphi}^\dagger \partial^\mu \hat{\varphi} - \hat{\varphi}\, \partial \hat{\varphi}^\dagger:$$

The constant of the motion comes from

$$J^0(x) = i: \varphi^\dagger \dot{\varphi} - \varphi\, \dot{\varphi}^\dagger:$$

Call the "charge" associated with this $U(1)$ symmetry "T":

$$\left.\begin{aligned} T &= \int d^3x\, J^0(x) \\ &= i \int d^3x\, :\varphi^\dagger \dot{\varphi} - \varphi\, \dot{\varphi}^\dagger: \end{aligned}\right\} \quad U(\alpha) = e^{i\alpha T}$$

This is like all of the others that we've done.. the result is

$$T = \int dK \left[a^\dagger(k) a(k) - b^\dagger(k) b(k) \right]$$
$$= \int dK \left[N_a(k) - N_b(k) \right]$$

→ T is an operator, the generator of this $U(1)$ transformation

note: · there is no weighting of this integral by anything dimensionful -- like $\hbar\omega$ ⇒ regardless of the kinematics of the φ, each quantum counts the same

→ such a characteristic is called "internal"

The application of T to a state counts the net amount of this characteristic and labels the fields created by $a^\dagger$ with the opposite sign but same magnitude as those created by $b^\dagger$.

Remember: before we did the transformation as

$$\varphi \rightarrow \varphi' = e^{i\alpha T} \varphi$$

<u>but</u> now φ is an operator and that's not how operators transform. Rather,

$$\varphi \rightarrow \varphi' = U(\alpha)\, \varphi\, U^{-1}(\alpha)$$

$$\varphi' = e^{i\alpha T}\,\varphi\,e^{-i\alpha T}$$
$$= (1 + i\alpha T + \tfrac{1}{2}(i\alpha T)^2 + \cdots)\,\varphi\,(1 - i\alpha T + \tfrac{1}{2}(-i\alpha T)^2 + \cdots)$$
$$= \varphi + i\alpha(T\varphi - \varphi T) + \tfrac{1}{2}(i\alpha)^2(TT\varphi - 2T\varphi T) + \cdots$$

$$\varphi' = \varphi + i\alpha[T,\varphi] + \tfrac{1}{2}(i\alpha)^2[T,[T,\varphi]] + \cdots$$

so, we need $[T,\varphi]$

$$[T,\varphi] = \int dK' \int dK \left[a^+(k)c(k) - b^+(k)b(k)\,,\ a(k')e^{-ik'x} + d^+(k')e^{ik'x}\right]$$

⋮ problem assignment

$[T,\varphi] = -\varphi$ and $[T,\varphi^+] = \varphi^+$

also, $[T,a] = -a$ $\quad [T,b^+] = -b^+$
$[T,a^+] = a^+$ $\quad [T,b] = b$

so $\varphi' = \varphi + i\alpha(-\varphi) + \tfrac{1}{2}(i\alpha)^2[T,-\varphi]\cdots$
$= \varphi - i\alpha\varphi - \tfrac{1}{2}\alpha^2\varphi \cdots$

$\varphi' = e^{-i\alpha}\varphi$ showing that for $U(1)$ this is still a simple phase rotation.

Suppose we have

$$|\psi\rangle = |n_a(k_1), n_a(k_2)\cdots, n(k_i), \cdots; n_b(k_1), n_b(k_2)\cdots n_b(k_i)\cdots\rangle$$

$$T|\psi\rangle = \int dK\ [N_a(k) - N_b(k)]\,|\psi\rangle$$

$$= \int dK\ \underbrace{(\ n_a(k) - n_b(k))}_{\text{net } \#}\,|\psi\rangle$$

how about $\quad T\,a^{+}(k_i)|\psi\rangle$

$$T\,a^{+}(k_i)|\psi\rangle = \int dK\ [N_a(k) - N_b(k)]\,a^{+}(k)\,|\psi\rangle$$

$$= \int dK\ [\ N_a(k) - N_b(k)]\sqrt{n(k_i)}\,|\ldots, n_a(k_i)+1, \ldots; \ldots n_b(k_i)\ldots\rangle$$

not useful, or an eigenvalue equation.

Instead use commutator,

$$[T, a^{+}(k_i)] = +a^{+}(k_i)$$

$$T a^{+}(k_i) - a^{+}(k_i)\,T = +a^{+}(k_i)$$

$$T a^{+}(k_i) = a^{+}(k_i)\,T + a^{+}(k_i)$$

$$T\,a^{+}(k_i)|\psi\rangle = (\,a^{+}T + a^{+}(k_i)\,)\,|\psi\rangle$$

$$= a^{+}(k_i)\int dK\ [\ N_a - N_b]\,|\psi\rangle + a^{+}(k_i)|\psi\rangle$$

$$= a^{+}(k_i)\int dK\ (\ n_a - n_b)\,|\psi\rangle + a^{+}(k_i)|\psi\rangle$$

$$= \int dK\,(n_a - n_b)\,a^{+}(k_i)|\psi\rangle + a^{+}(k_i)|\psi\rangle$$

$$= \left\{\int dK\ [n_a(k) - n_b(k)] + 1\right\} a^{+}(k_i)\,|\psi\rangle$$

so $\quad a^{+}(k_i)|\psi\rangle \quad$ contains 1 more quantum of charge +1 than $|\psi\rangle$

likewise $a(k)|\Psi\rangle$ contains 1 less.

Also, $T\, b^{+}|\Psi\rangle = \left\{ \int dK \left[n_a(k) - n_b(k) \right] - 1 \right\} b^{+}|\Psi\rangle$

so this supports the idea of antiparticles having opposite internal quantum characteristics ("charges").

$a^{+}(\vec{k})$	creates a spin 0 particle of	$\vec{k}$ and	$T = +1$
$b^{+}(\vec{k})$	" " " " " "	$\vec{k}$ "	$T = -1$
$a(\vec{k})$	annihilates "	$\vec{k}$ "	$T = +1$
$b(\vec{k})$	" "	$\vec{k}$ "	$T = -1$

in our "frequency" description

$$\varphi = \underbrace{\varphi^{(+)}}_{\substack{\text{annihilates} \\ \text{particle}}} + \underbrace{\varphi^{(-)}}_{\substack{\text{creates} \\ \text{antiparticle.}}}$$

$$\varphi^{+} = \underbrace{\varphi^{(+)+}}_{\substack{\text{creates} \\ \text{particle}}} + \underbrace{\varphi^{(-)+}}_{\substack{\text{annihilates} \\ \text{antiparticle}}}$$

Recall,

$$\mathcal{L}(x) = \bar{\psi}(x)\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x)$$

which gave $\quad \mathcal{H} = \pi\dot{\psi} - \mathcal{L}$

$$\pi = i\psi^+$$

$$\mathcal{H} = i\psi^+ \partial^0\psi - \bar{\psi}\left(i\gamma^0\partial_0 + i\gamma^i\partial_i - m\right)\psi$$
$$= i\bar{\psi}\gamma^0\partial_t\psi - i\bar{\psi}\gamma^0\partial_t\psi - i\bar{\psi}\gamma^i\partial_i + \bar{\psi}m\psi$$
$$= \psi^+\gamma^0(-i\gamma^i\partial_i + m)\psi$$

from Dirac equation, the stuff in $( ) = i\gamma^0\partial_t$

$$\mathcal{H} = \psi^+\gamma^0(i\gamma^0\partial_t\psi)$$
$$\mathcal{H} = i\psi^+\frac{\partial\psi}{\partial t}$$

Quantization proceeds as usual, with an important exception.

Expand, component by component,

$$\psi_j(x) = \sum_{i=1,2}\int dK\left[\underbrace{a^{(i)}(h)u_j^{(i)}(h)e^{-ih\cdot x}}_{\psi^{(+)}} + \underbrace{b^{(i)+}(h)v_j^{(i)}(h)e^{ih\cdot x}}_{\psi^{(-)}}\right]$$

where

$$u^{(i)}(k) = \sqrt{E+m}\begin{pmatrix} \chi^{(i)} \\ \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\,\chi^{(i)} \end{pmatrix}$$

$i=1 \Rightarrow$ "spin" $+\frac{1}{2}$

$=2 \Rightarrow$ $-\frac{1}{2}$

$\chi^{(1)} = \binom{1}{0}$ $\chi^{(2)} = \binom{0}{1}$

$$v^{(i)}(k) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{k}}{E+m}\,\chi^{(i)} \\ \chi^{(i)} \end{pmatrix}$$

$i=1 \Rightarrow$ "spin" $-\frac{1}{2}$

$=2 \Rightarrow$ $+\frac{1}{2}$

and $u^{+(i)}u^{(j)} = 2E\,\delta_{ij} \Rightarrow \bar{u}^{(i)}u^{(j)} = 2m\,\delta_{ij}$

$v^{+(i)}v^{(j)} = 2E\,\delta_{ij}$ $\quad \bar{v}^{(i)}v^{(j)} = -2m\,\delta_{ij}$

plus $(\not{k}-m)\,u(k) = 0$ $\quad \bar{u}(k)(\not{k}-m) = 0$

$(\not{k}+m)\,v(k) = 0$ $\quad \bar{v}(k)(\not{k}+m) = 0$

so,

$$\bar{\psi}_j(x) = \sum_{i=1,2}\int dK\left[\bar{u}_j^{(i)}(k)\,a^{(i)+}(k)\,e^{ik\cdot x} + \bar{v}_j^{(i)}\,b^{(i)}(k)\,e^{-ik\cdot x}\right]$$

$$\bar{\psi}^{(-)} + \bar{\psi}^{(+)}$$

The Hamiltonian provides the surprise.-

$$H = \int d^3x\,\mathcal{H}(x)$$

$$= \int d^3x\, i\,\psi^+\frac{\partial\psi}{\partial x^0} \quad \leftarrow \text{matrix equation.}$$

$$= i\int d^3x \sum_{i=1,2}\sum_{n=1,2}\int\frac{d^3k}{(2\pi)^3 2E}\int\frac{d^3k'}{(2\pi)^3 2E'}\left\{\left[\psi^+\right]\frac{\partial}{\partial t}\left[\psi\right]\right\}$$

$$\sum_i \sum_n \{\ \} = \sum_i \sum_n \left[a^{+(n)} u^{+(n)} e^{ih\cdot x} + b^{(i)} v^{(i)+} e^{-ih\cdot x} \right]$$

$$\times \left[(-iE') a^{(n)} u^{(n)} e^{-ih'\cdot x} + (iE') b^{+(n)} v^{(n)} e^{ih'\cdot x} \right]$$

$\therefore$ problem.

$$H = \int dK \sum_i E \left[a^{+(i)}(k) a^{(i)}(k) - b^{(i)}(k) b^{+(i)}(k) \right]$$

care is now required!

Suppose : $[b, b^+] = c\,\delta(\ )$ like spin ϕ, then
$$b b^+ = b^+ b + c\,\delta(0)$$

substitute as before,
$$[\, a^+ a - b^+ b - c\,\delta(0) \,]$$

get rid of it

BUT $\quad - b^+ b$ would still result in non positive definite energy eigenvalues of H.

must postulate anticommutators –

$$\{ a^{(i)}(k), a^{+(j)}(k') \} = \{ b^{(i)}(k), b^{+(j)}(k') \} = (2\pi)^3 2E \delta_{ij} \delta^3(k-k')$$

$$\{b, b'\} = \{a, a'\} = \{a^+, a^+\} = \{b^+, b^+\} = 0$$

Then we have a sensible theory – with a new Normal ordering

$H = \int dK \; : i \psi^\dagger \partial_t \psi :$

which amounts to using $\{\}$ to $\quad \{b, b^\dagger\} = i\delta(\,)$

$\qquad bb^\dagger = -b^\dagger b + i\delta(\,)$

$$a^\dagger a - b\, b^\dagger \longrightarrow a^\dagger a + b^\dagger b - i\delta(0)$$

ah. $\quad$ forget it — just like before

so, now spin 0 & $\frac{1}{2}$: $\quad$ P = # fermion permutations

$$: ABCD : \equiv (-1)^P [\underbrace{---}]$$

ABCD arranged: creation - LEFT
annihilation - RIGHT

so, $\quad : \psi \bar{\psi}' : = : (\psi^+ + \psi^-)(\bar{\psi}^+ + \bar{\psi}^-)$

$$= : \underset{a}{\psi^+} \underset{b}{\bar{\psi}^+} + \underset{a}{\psi^+} \underset{a^+}{\bar{\psi}^-} + \underset{b^+}{\psi^-} \underset{b}{\bar{\psi}^+} + \underset{b^+}{\psi^-} \underset{a^+}{\bar{\psi}^-} :$$

$$= \psi^+ \bar{\psi}^+ - \bar{\psi}^- \psi^+ + \psi^- \bar{\psi}^+ + \psi^- \bar{\psi}^-$$

Can get the commutation relations for field operators by, like before, inverting the expansions to get a and b..

form $\quad u_j^{(\mu)\dagger}(k')\, e^{i k' \cdot x} \cdot \psi_j(x)$

integrate $\int d^3x$ and $\sum_j$

$$\sum_j \int d^3x \, e^{ik'\cdot x} u_j^{(n)\dagger}(k') \psi_j(x) =$$

$$\int d^3x \sum_j \sum_i dK \Big\{ \big(a^i(k)\, u_j^{(n)\dagger}(k')\, u_j^{(i)}(k)\big) e^{i(k'-k)\cdot x} + \big(b^{(i)\dagger}(k)\, u_j^{(n)\dagger}(k')\, v_j^{(i)}(k)\big) e^{i(k'+k)\cdot x} \Big\}$$

$$= \sum_i \sum_j \int dK \Big\{ (\quad)(2\pi)^3 \delta(\vec{k}-\vec{k}')\, e^{i(E'-E)t} + (\quad)(2\pi)^3 \delta(\vec{k}+\vec{k}') \Big\} e^{i(E'+E)t} \Big\}$$

use orthogonality $= 2E\,\delta_{ni}$

use $\delta(k-k')$ & $E^2 = k^2 + m^2$ $\Big\}$ 1

use orth $\Rightarrow \emptyset$

so, all collapses to

$$= a^{(n)}(k)$$

likewise,

$$a^{(n)\dagger}(k) = \sum_j \int d^3x \, e^{-ik\cdot x} \psi_j^\dagger u_j^{(n)}(k)$$

$$b^{(n)\dagger}(k) = \sum_j \int d^3x \, e^{-ik\cdot x} v_j^{(n)\dagger}(k) \psi_j(x)$$

$$b^{(n)}(k) = \sum_j \int d^3x \, e^{ik\cdot x} \psi_j^\dagger(x) v_j^{(n)}(k)$$

So, the required anticommutation relations for a's, b's result in

$$\{ \psi_i(\vec{x},t), \psi_j^+(\vec{x}',t)\} = \delta_{ij}\, \delta(\vec{x}-\vec{x}')$$

$$\{ \psi, \psi\} = \{ \psi^+, \psi^+\} = 0$$

So, were now too deep in to back out now! We must reinterpret our 2nd quantization conditions to mean

$$[X, \Pi'] = i\, \delta(x-x') \qquad \text{bosons.}$$

$$\{ X, \Pi \} = i\, \delta(x-x') \qquad \text{fermions.}$$

same physical interpretation:

a^+ and b^+ create a & b- type quanta (fermion & antifermion)
a and b annihilate them

The charge operator becomes.

$$T = \int d^3x \, :\bar{\psi}\gamma^0\psi: \qquad \text{remember } j^\mu = \bar{\psi}\gamma^\mu\psi$$
$$= \int d^3x \, :\psi^+\psi:$$
$$= \int dK \sum_i [a^{(i)+}(k)\, a^{(i)}(k) - b^{(i)+}(k)\, b^{(i)}(k)]$$

b- type has opposite "charge" from a- type.

Consider a state like $\Psi = \begin{pmatrix} p \\ n \end{pmatrix}$ an isodoublet

$$\mathcal{L} = \bar{\Psi}\,(i\gamma^\mu \partial_\mu - m)\,\Psi$$

investigate the consequences of the $U(1)$ transformation

$$\Psi \to \Psi' = e^{i\alpha}\Psi$$ .. on both p and n states.

The Noether current is still of the form

$$j^\mu(x) = -i \frac{\partial \mathcal{L}}{\partial(\partial\Psi/\partial x^\mu)}\,\Psi = \bar{\Psi}\gamma^\mu\Psi$$

charge
$$B = \int d^3x \,:\Psi^\dagger\Psi:$$
$$= \int d^3x \,:(p^\dagger n^\dagger)\begin{pmatrix} p \\ n \end{pmatrix}:$$
$$= \int d^3x \,: p^\dagger p + n^\dagger n :$$

p and n are each separate spinors, each with its own expansion,

$$B = \int d^3x \left[N_p(x) + N_n(x) - N_{\bar{p}}(x) - N_{\bar{n}}(x) \right]$$

net B charge .. obviously not electric charge, but rather Baryon Number charge, a quantity conserved in elementary particle physics reactions ie, you never see

$$p \to e^+ \nu_e \quad \text{etc.}$$