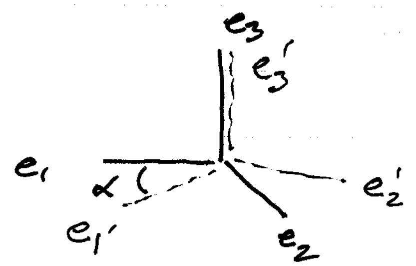
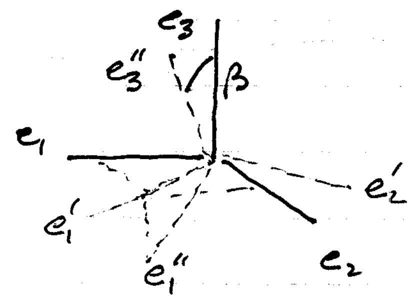
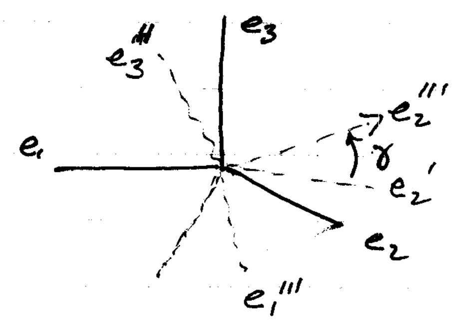
The standard way to represent 3d rotations is in terms of successive 2d rotations... the Euler Angles.



α about e_3



β about e_2'



γ about e_3'''

$$SO(3) \rightarrow 3\ SO(2)$$

$$\mathbb{R}(\alpha,\beta,\gamma) = \mathbb{R}_{e_3''}(\gamma)\,\mathbb{R}_{e_2'}(\beta)\,\mathbb{R}_{e_3}(\alpha)$$

wrt angles attached to the rotating object.

More convenient are to fixed axes:

$$\mathbb{R}_{e_3}(\alpha) = \mathbb{R}_3(\alpha)$$

$$\mathbb{R}_{e_2'}(\beta) = \mathbb{R}_3(\alpha)\mathbb{R}_2(\beta)\mathbb{R}_3(-\alpha)$$

$$\mathbb{R}_{e_3''}(\gamma) = \mathbb{R}_3(\alpha)\mathbb{R}_2(\beta)\mathbb{R}_3(\gamma)\mathbb{R}_2(-\beta)\mathbb{R}_3(-\alpha)$$

Substituting

$$\mathbb{R}(\alpha,\beta,\gamma) = \mathbb{R}_3(\alpha)\mathbb{R}_2(\beta)\mathbb{R}_3(\gamma)\mathbb{R}_2(-\beta)\mathbb{R}_3(-\alpha)\mathbb{R}_3(\alpha)\mathbb{R}_2(\beta)\mathbb{R}_3(-\alpha)\mathbb{R}_3(\alpha)$$

$$= \mathbb{R}_3(\alpha)\mathbb{R}_2(\beta)\mathbb{R}_3(\gamma)$$

where, as before, $\mathbb{R}_i(\theta) = e^{-i J_i \theta}$

In QM.

The wavefunction in configuration space.

$$\psi(\vec{x}) = \langle \vec{x} | \psi \rangle$$

representing same $R(\alpha, \beta, \gamma)$

if rotate coordinates $|\vec{x}\rangle \Rightarrow |\vec{x}'\rangle = \Gamma(R)|\vec{x}\rangle$

we'll change the wavefunction $\psi(x') \to \psi'(x)$

Our procedure for this:

$$|\psi\rangle = \int |\vec{x}\rangle \psi(\vec{x})\, d\vec{x}$$

expand complete set.

project against transformed coordinates.

$$\langle \vec{x}' | \psi \rangle = \int \langle \vec{x}' | \vec{x} \rangle \psi(\vec{x})\, d\vec{x}$$
$$= \int \delta(\vec{x} - \vec{x}')\psi(\vec{x})\, d\vec{x} = \psi(\vec{x}') = \psi(R^{-1}\vec{x})$$

can also represent this same transformation using a Hilbert Space representation.

$$|\psi\rangle \longrightarrow |\psi'\rangle = \mathcal{R}|\psi\rangle$$

H.S. operator.

$H\psi = E\psi$

~~$H\psi$~~

if ~~$H\mathcal{R}\psi = E\mathcal{R}\psi = E\psi'$~~

~~$\mathcal{R}H\psi =$~~

$$H\psi = E\psi$$

$R \rightarrow \qquad R H\psi = E\, R\psi$

if $\quad [R, H] = 0 \quad \Rightarrow \quad H R\psi = E R\psi$

$$H\psi' = E\psi' \quad < \text{degenerate}$$

Then, this implies $\quad [H, J_\sigma] = 0 \qquad \sigma = 1,2,3$

for

$$R_n = e^{-i\vec{J}\cdot\hat{n}\theta}$$

so angular momentum conservation comes from invariance w/ respect to rotation.

So, under these circumstances, the $|\psi^*\rangle$ all transform among themselves through R. $\rightarrow$ a subspace of all H.S. vectors... built from angular momentum eigenstates.

Label states by a pair of indices:

for n^{th} IRR of SO(3), count the basis vectors by m

$$|\psi_m^{(n)}\rangle \qquad (\text{think } l,m: \psi^{(1)}_{-1,0,1})$$

Then the H.S. action of the SO(3) transformation does this:

$$|\psi_i^{(n)}\rangle \longrightarrow |\psi_i^{(n)\prime}\rangle = |\psi_j^{(n)}\rangle \underbrace{D^{(n)}(R)^j{}_i}_{\text{a matrix rep. of } R}$$

of J_1, J_2, J_3, it's customary to label the weights according to J_3.

$$J_3 \,|\, \xi_m^{(n)} \rangle = m \,|\, \xi_m^{(n)} \rangle$$

$\xi_m^{(n)}$ are not eigenstates of J_1 or J_2 and that comes from the SO(3) Lie Algebra.

Further, there is always a special operator – called Casimir Operator of which $\xi_n^{(m)}$ is an eigenstate

$$J^2 = J_1^2 + J_2^2 + J_3^2$$

$$J^2 \,|\, \xi_m^{(n)} \rangle = |\, \xi_m^{(n)} \rangle \, n(n+1)$$

does not depend on weight.

a whole tensor analysis can result ...

back to D's

$$|\psi_m^{(j)}\rangle' = R(\alpha,\beta,\gamma) \,|\, \psi_m^{(j)} \rangle = |\psi_n^{(j)}\rangle \, D^{(j)}(\alpha,\beta,\gamma)^n{}_m$$

matrix element

$$D^{(j)}(\alpha\beta\gamma)^n{}_m = \langle \psi^{n(j)} \,|\, R(\alpha\beta\gamma) \,|\, \psi_m^{(j)} \rangle$$

are the "Wigner Functions"

In the representation in which J_3 is diagonal, use Euler angles

$$R(\alpha\beta\gamma) = e^{-i\alpha J_3} e^{-i\beta J_2} e^{-i\gamma J_3}$$

$$e^{i\gamma J_3}|\psi_m^{(j)}\rangle = \sum_{n=0}^{\infty} \frac{(-i)^n(\gamma)^n}{n!} J_3^n |\psi_m^{(j)}\rangle$$ Maclaurin exp. ~~MED~~

$$= \sum_{n=0}^{\infty} \frac{(-i)^n(\gamma)^n}{n!} m^n |\psi_m^{(j)}\rangle = e^{-i\gamma m}|\psi_m^{(j)}\rangle$$

so,

$$D^{(j)}(\alpha,\beta,\gamma)^n{}_m = e^{-i\alpha n}\, d^{(j)}(\beta)^n{}_m\, e^{-i\gamma m}$$

where $$d^{(j)}(\beta)^n{}_m = \langle \psi^{(j)n}| e^{-i\beta J_2} |\psi_m^{(j)}\rangle$$

called the "little d's"

Easy to construct. For spin 1/2...

$$d^{(1/2)}(\beta) = e^{-i\beta/2\,\sigma_2} = \mathbb{1}\cos\beta/2 - i\sigma_2 \sin\beta/2$$
$$= \begin{pmatrix} \cos\beta/2 & -\sin\beta/2 \\ \sin\beta/2 & \cos\beta/2 \end{pmatrix}$$

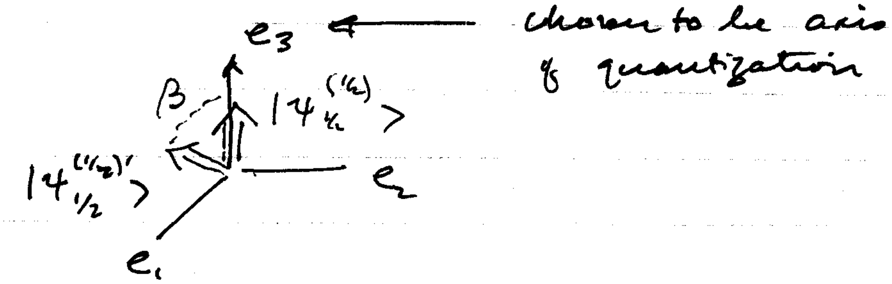
and element by element...

$$d^{(1/2)}(\beta)^{1/2}{}_{1/2} = (1,0)\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c$$

$$= \cos\beta/2$$

etc.

They can be used to rotate spinors.



$$R(0,\beta,0)\,|\psi_{1/2}^{(1/2)}\rangle = |\psi_n^{(1/2)}\rangle\, D^{(1/2)}(0,\beta,0)^n{}_{1/2}$$

$$= |\psi_{-1/2}^{(1/2)}\rangle D^{(1/2)}(0,\beta,0)^{-1/2}{}_{1/2} + |\psi_{1/2}^{(1/2)}\rangle D^{(1/2)}(0,\beta,0)^{1/2}{}_{1/2}$$

$$= \begin{pmatrix}0\\1\end{pmatrix}\sin\beta/2 + \begin{pmatrix}1\\0\end{pmatrix}\cos\beta/2$$

$$= \begin{pmatrix}\cos\beta/2\\ \sin\beta/2\end{pmatrix}$$

The $j=1$ d's are not-easily constructed...

$$d^{(1)}(\beta) = \begin{pmatrix}\frac{1+\cos\beta}{2} & - & - \\ -1 & - & \\ - & & - \end{pmatrix}$$ no half-angle rotations.

D's all all matrix representations of SU(2)
$D^{(n)}$'s (n = integer) are matrix rep of SO(3)

} "Rotation group"