Lecture 6 2/2/04 SU(2)SO(3) 79

Another group... one that does this to vectors.

$$|\xi\rangle \longrightarrow |\xi'\rangle = R(\vec{\alpha})|\xi\rangle$$
$$|e_i\rangle \xi^{i'} = |e_i\rangle R(\alpha)^i{}_j \xi^j$$

thereby transforming the components: $\xi^{i'} = R(\alpha)^i{}_j \xi^j$

Insist that

① Unitary: $RR^+ = R^+R = 1$ — let inverse $A = R^{-1} = R^+$

then $RA = \mathbb{1}$

$R^i{}_j A^j{}_k = \delta^i{}_k$

But $A^j{}_k = R^{+j}{}_k = R^*{}_k{}^j = (R^k{}_j)^*$

so,

$$RR^+ = 1 \Rightarrow \sum_j R^i{}_j R^{+j}{}_k = \delta^i{}_k = R^i{}_j R^*{}_k{}^j$$

② Real:
$$\sum_j R^i{}_j R_k{}^j = \delta^i{}_k$$

$$\sum_j R^i{}_j (R^j{}_k)^T = \delta^i{}_k$$

$$\mathbb{R}\mathbb{R}^T = \mathbb{1}$$

③ length - norm - preserving for the basis vectors.

$$|e_i'\rangle = |e_j\rangle R^j{}_i \qquad \left(\Rightarrow \quad \langle e^{i'}| = R^{+i}{}_j \langle e^j| \right)$$

denote the inverse $\quad |e_i\rangle = |e_j'\rangle S^j{}_i$

from the scalar product

$$\langle e^{k'}|e_i\rangle = R^{+k}{}_l \langle e^l|e_i\rangle = R^{+k}{}_l \delta^l{}_i = R^{+k}{}_i$$
$$= \langle e^{k'}|e_j'\rangle S^j{}_i = \delta^k{}_j S^j{}_i = S^k{}_i$$

so, $\quad S^k{}_i = R^{+k}{}_i = R^*{}_i{}^k = (R^i{}_k)^*$

the bases are real $\Rightarrow$ the matrices are real

$$S^k{}_i = R^i{}_k \Rightarrow (R^{-1})^i{}_k = R^k{}_i \Rightarrow \mathbb{R}^{-1} = \mathbb{R}^T$$

Also, $\langle e^{k\prime} | e_i{}' \rangle = R^{\dagger k}{}_l \langle e^l | e_j \rangle R^j{}_i$

$\delta^k{}_i = R^{\dagger k}{}_l R^j{}_i \delta^l{}_j = R^{\dagger k}{}_j R^j{}_i$

so R is orthogonal which then makes

$\langle e^{k\prime} | e_i{}' \rangle = \langle e^k | e_i \rangle \Rightarrow$ length-preserving.

Finally, this insures.

$R^{\dagger k}{}_j R^j{}_i = \delta^k{}_i$ orthogonality

$= R^*{}_j{}^k R^j{}_i = (R^j{}_k)^* R^j{}_i = R^j{}_k R^i{}_j = \delta^k{}_i$ real

$\det(R^j{}_k R^j{}_i) = \det \delta^k{}_i = 1$ since $k=i$

so

$$[\det(R^i{}_i)]^2 = 1 \Rightarrow \det \mathbb{R} = \pm 1$$

So, we can have two different sets

$\det \mathbb{R} = +1$ group of "proper" rotations

$\det \mathbb{R} = -1$ don't actually have a group -- "improper" rotations

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As before.. prepare for infinitesimal transformations by carefully identifying the parameter corresponding to the <u>identity</u>.

Define $\quad \mathbb{R} = \mathbb{1} + \mathbb{N}$

so, near the identity $\quad \mathbb{R} = \mathbb{1} + \delta\mathbb{N} = \mathbb{1} + \eta$

Orthogonality says $\quad R^j{}_k R^j{}_i = \delta_{ki}$

so,

$$(\delta^j{}_k + \eta^j{}_k)(\delta^j{}_i + \eta^j{}_i) = \delta_{ki}$$

$$\delta^j{}_k \delta^j{}_i + \delta^j{}_k \eta^j{}_i + \eta^j{}_k \delta^j{}_i + \eta^2 =$$

$$\delta^k{}_i + \eta^k{}_i + \eta^i{}_k = \delta_{ki}$$

so,

$$\eta^k{}_i = -\eta^i{}_k \Rightarrow \eta = -\eta^T \Rightarrow \eta^i{}_i = 0$$

Then,

$$\xi^i \to \xi^{i\prime} = (\delta^i{}_j + \eta^i{}_j)\xi^j$$ is the infinitesimal transformation

$$\xi^{i\prime} = \xi^i + \delta\xi^i = \delta^i{}_j \xi^j + \eta^i{}_j \xi^j$$

$$\delta\xi^i = \eta^i{}_j \xi^j$$

arbitrary dimensions.. consider 3d..

$$\delta\xi^1 = \cancel{\eta^1{}_1 \xi^1} + \eta^1{}_2 \xi^2 + \eta^1{}_3 \xi^3$$

$$\delta\xi^2 = \eta^2{}_1 \xi^1 + \cancel{\eta^2{}_2 \xi^2} + \eta^2{}_3 \xi^3$$

$$\delta\xi^3 = \eta^3{}_1 \xi^1 + \eta^3{}_2 \xi^2 + \cancel{\eta^3{}_3 \xi^3}$$

Recast this in terms of one index -- and define the parameters namely, for ijk cyclic, 1,2,3

call $\eta^i{}_j = \delta\alpha_k$ so

$$\alpha = \begin{pmatrix} 0 & \delta\alpha_3 & -\delta\alpha_2 \\ -\delta\alpha_3 & 0 & \delta\alpha_1 \\ \delta\alpha_2 & -\delta\alpha_3 & 0 \end{pmatrix}$$

and we can write

$$\left.\begin{aligned} \delta\xi^1 &= \delta\alpha_3\,\xi^2 - \delta\alpha_2\,\xi^3 \\ \delta\xi^2 &= -\delta\alpha_3\,\xi^1 + \delta\alpha_1\,\xi^3 \\ \delta\xi^3 &= \delta\alpha_2\,\xi^1 - \delta\alpha_3\,\xi^2 \end{aligned}\right\} \quad \begin{aligned} &\delta\xi^i = \xi^j\alpha_k - \xi^k\alpha_j \\ &i \neq j \neq k \end{aligned}$$

Now. For the general form

$$\delta\xi^i = u^i{}_\sigma\,\delta\alpha^\sigma$$

to pick off

$$\delta\xi^1 = u^1{}_1\,\delta\alpha^1 + u^1{}_2\,\delta\alpha_2 + u^1{}_3\,\delta\alpha^3$$

so

$$u^1{}_1 = 0\,; \quad u^1{}_2 = -\xi^3 = -u^2{}_1$$
$$u^1{}_3 = \xi^2 = -u^3{}_1$$

and we can construct the generators

$$X_\sigma = u^i{}_\sigma \frac{\partial}{\partial \xi^i}$$

$$X_1 = u^1{}_1 \frac{\partial}{\partial \xi^1} + u^2{}_1 \frac{\partial}{\partial \xi^2} + u^3{}_1 \frac{\partial}{\partial \xi^3}$$

$$X_1 = \xi^3 \frac{\partial}{\partial \xi^1} - \xi^2 \frac{\partial}{\partial \xi^3} \qquad \text{etc.}$$

and the algebra can be worked out as before..

$$[X_1, X_2] = X_3$$
$$[X_2, X_3] = X_1$$
$$[X_3, X_1] = X_2$$

Conventionally, in physics $J_\sigma \equiv i X_\sigma$

so,

$$[J_\sigma, J_\rho] = i\, \varepsilon_{\sigma\rho\pi} J^\pi$$

where $\varepsilon_{\sigma\rho\pi}$ is the antisymmetric tensor

→ the angular momentum operator algebra.

Group $\quad SO(3)$

"special" $\det \mathbb{R} = +1$

"orthogonal"

3 dimensions.

leaving in $\det \mathbb{R} = \pm 1 \Rightarrow$ larger group $O(3)$

rotations <u>and</u> inversions $\quad I_s \vec{x} = -\vec{x}$

The inversion group is called J so,

$$O(3) = SO(3) \otimes J$$

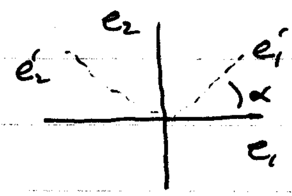
Remember, for infinitesimal

$$f(\vec{\xi}') = f(\vec{\xi}) + df(\vec{\xi})$$

$$df = f(\xi') - f(\xi) = \delta\alpha^\sigma X_\sigma f(\xi)$$

$$X_\sigma f(\vec{\xi}) = \lim_{\delta\alpha \to 0} \frac{f(\vec{\xi}') - f(\vec{\xi})}{\delta\alpha^\sigma}$$

2d .. $SO(2)$



$$|e_i'\rangle = |e_j\rangle R^i{}_j$$

and $\mathbb{R} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ and components $x'^i = R^i{}_j x^j$

$\cancel{R = (\cos\alpha}$

Consider a very small angle

$$f(\vec{x}') = f(x', y')$$
$$= f(x\cos\delta\alpha - y\sin\delta\alpha,\ x\sin\delta\alpha + y\cos\delta\alpha)$$

Taylor-expand around identity

$$f(x', y') = f(x, y) - (x - x')\frac{\partial f}{\partial x} - (y - y')\frac{\partial f}{\partial y} + \dots$$

in limit, can write

$$X f(\vec{x}) = \lim_{\delta\alpha \to 0} \frac{-(x - x')\partial f/\partial x - (y - y')\partial f/\partial y}{\delta\alpha}$$

$$=$$

$$= \lim_{\delta\alpha\to 0} \frac{\left[-(x - x\cos\delta\alpha + y\sin\delta\alpha)\frac{\partial f}{\partial x} - (y - x\sin\delta\alpha - y\cos\delta\alpha)\frac{\partial f}{\partial y}\right]}{\delta\alpha}$$

$$= \frac{-(x - x + y\,\delta\alpha)\frac{\partial f}{\partial x} - (y - x\,\delta\alpha - y)\frac{\partial f}{\partial y}\ ]}{\delta\alpha}$$

$$X f = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) f$$

which is the X_3 or J_3 generator of $SO(3)$

ie

$$f(\vec{x}) \to f(\vec{x}\,') = \underbrace{\mathbb{R}_3}_{\text{around 3-axis}}(\delta\alpha)\, f(\vec{x}) \qquad \text{written as infinitesimal}$$

$$= (+\,\delta\alpha^3 X_3)\, f(\vec{x})$$

Imagine rotating by finite amount α^3, then $\delta\alpha^3$

$$\mathbb{R}_3(\delta\alpha)\mathbb{R}(\alpha^3) f(\vec{x}) = (1 + \delta\alpha^3 X_3)\,\mathbb{R}_3(\alpha^3)\, f(\vec{x})$$

$$= \mathbb{R}_3(\alpha^3) f(\vec{x}) + \delta\alpha^3 X_3\, \mathbb{R}_3(\alpha^3) f(\vec{x})$$

$$= \mathbb{R}_3(\alpha^3 + \delta\alpha^3) f(\vec{x}) \qquad \text{from group property}$$

so,

$$X_3\,\mathbb{R}_3(\alpha^3) = \frac{\mathbb{R}_3(\alpha^3 + \delta\alpha^3) - \mathbb{R}(\alpha^3)}{\delta\alpha^3}$$

$$\lim_{\delta\alpha^3\to 0} \quad X_3\,\mathbb{R}_3(\alpha^3) = \frac{d\,\mathbb{R}_3(\alpha)}{d\alpha^3}$$

so formally,

$$\mathbb{R}_3(\alpha^3) = e^{\alpha^3 J_3} \qquad \text{for finite angle}$$

So this gives another way to write the generators in terms of infinitesimal matrices

$$J_3 = i\frac{\partial \mathbb{R}_3(\alpha^3)}{\partial \alpha^3}\frac{1}{\mathbb{R}_3(\alpha^3)}\Bigg|_{\alpha^3\to 0} = i\frac{\partial \mathbb{R}_3(\alpha^3)}{\partial \alpha^3}$$

So, $$\mathbb{R}_3(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$i\frac{\partial \mathbb{R}_3}{\partial \alpha} = \begin{pmatrix} -i\sin\alpha & -i\cos\alpha & 0 \\ i\cos\alpha & -i\sin\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lim_{\alpha\to 0} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

hmm.

Another group.

transformation of a vector $\quad \xi \to \xi' = u(\alpha)^i{}_j\, \xi^j$

when the matrices are

unitary, unimodular — but not necessarily real.

Specify 2 dimensions. Then one can show that the most general form of a matrix... is

$$u = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \qquad \left.\begin{aligned} u^\dagger u &= 1 \\ \det u &= +1 \end{aligned}\right\} \; SU(2).$$

$\leftarrow$ "special"

Again, $\quad u = \mathbb{1} + \mathfrak{u}$

$\uparrow$ infinitesimal.

$$\begin{aligned} \xi^{i\prime} &= (\delta^i{}_j + u^i{}_j)\,\xi^j \\ &= \xi^i + \underbrace{u^i{}_j\,\xi^j}_{\delta\xi^i} \end{aligned}$$

Three parameters are required (remember, can be complex)

$$\mathfrak{u} = \begin{pmatrix} -i\alpha_1 & \alpha_2 + i\alpha_3 \\ -\alpha_2 + i\alpha_3 & -i\alpha_1 \end{pmatrix}$$

⋮ same kind of steps.

$$X_\sigma = u^i{}_\sigma \frac{\partial}{\partial \xi^i} \longrightarrow [X_\sigma, X_\rho] = 2i\,\varepsilon_{\sigma\rho\pi} X^\pi$$

Define conventionally $\quad X_\sigma = -2i\,S_\sigma$

$$[S_\sigma, S_\rho] = i\,\varepsilon_{\sigma\rho\pi} S^\pi$$

— same as SO(3) algebra

Spin operators are defined $\quad S_p = \frac{1}{2}\sigma_p$

3 Pauli matrices. which satisfy the X_p algebra.

These are special matrices.

Consider any 2d matrix $\mathbb{H}$ with trace = 0. It can be represented in terms of a <u>3-dimensional</u> space vector:

$$\mathbb{H} = \vec{x}\cdot\vec{\sigma} = x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3$$

with matrix elements

$$h^i{}_j = (\vec{x}\cdot\vec{\sigma})^i{}_j$$

so
$$\mathbb{H} = \begin{pmatrix} x_3 & x_1 - i x_2 \\ x_1 + i x_2 & -x_3 \end{pmatrix}$$

So, transformations of $\vec{x} \to \vec{x}'$ could be done by transforming $\mathbb{H} \to \mathbb{H}'$ .. $\mathbb{H}$ can stand for $\vec{x}$.

Notice that

$$\det(\mathbb{H}) = -x_3^2 - (x_1 - i x_2)(x_1 + i x_2) = -x_1^2 - x_2^2 - x_3^2$$
$$= -(\text{length of } \vec{x})$$

To transform a matrix.. need some A such that

$$\mathbb{H}' = A\,\mathbb{H}\,A^{-1}$$

$\to$ restrict: 2d and hence ~~so~~ $SU(2)$ properties

So, $A = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$

$$\begin{pmatrix} x_3' & x_1' - i x_2' \\ x_1' + i x_2' & -x_3' \end{pmatrix} = H' = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} x^3 & x^1 - i x^2 \\ x^1 + i x^2 & -x^3 \end{pmatrix} \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix}$$

$$= \begin{pmatrix} aa^* x^3 + a^* b x^1 + i a^* b x^2 + a b^* x^1 - i a b^* x^2 - b b^* x^3 & \vdots \\ \vdots & \vdots \end{pmatrix}$$

etc.

Then, $x^3 \to x^{3'} = h^{1'} = \cdot$

$$= x^1 (a^* b + a b^*) + i x^2 (a^* b - a b^*) + x^3 (a a^* - b b^*)$$

which would be written as a 3x3 to reflect

$$\vec{x}' = R \vec{x}$$

where

$$R = \begin{pmatrix} a^* b + a b^* & i(a^* b - a b^*) & a a^* - b b^* \\ - & - & - \\ - & - & - \end{pmatrix}$$

if we completed this, we would find

R: real, orthogonal w/ $\det R = +1$

Since $\det H' = \det H$, so $|\vec{x}'| = |\vec{x}|$

it has the properties of a matrix representation of $SO(3)$!

What about the ξ? As always $\xi^{i\prime} = U(\vec{\alpha})^i{}_j \xi^j$.

Represent $\xi = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$ which is complex.

So, also have $\xi^\dagger \to \xi^{\dagger\prime} = \xi^\dagger U^\dagger$
which transform differently.

$$\begin{pmatrix} \xi^{1\prime} \\ \xi^{2\prime} \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = \begin{pmatrix} a\xi^1 + b\xi^2 \\ -b^*\xi^1 + a^*\xi^2 \end{pmatrix}$$

and

$$(\xi^{1*\prime}, \xi^{2*\prime}) = (\xi^{1*}, \xi^{2*})\begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix}$$

$$= (\xi^{1*}a^* + \xi^{2*}b^*, -b\xi^{1*} + a\xi^{2*})$$

So,

$$\xi^{1\prime} = a\xi^1 + b\xi^2 \quad \& \quad \xi^{2\prime} = -b^*\xi^1 + a^*\xi^2$$

$$-\xi^{2*\prime} = -a\xi^{2*} + b\xi^{1*} \quad \& \quad \xi^{1*\prime} = -b^*(-\xi^{2*}) + a^*\xi^{1*}$$

rename $-\xi^{2*} = \eta^1 \quad \xi^{1*} = \eta^2$
and

$$\begin{aligned} \eta^{1\prime} &= a\eta^1 + b\eta^2 \\ \eta^{2\prime} &= -b^*\eta^1 + a^*\eta^2 \end{aligned} \Rightarrow \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} \text{ transforms like } \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$$

or $\begin{pmatrix} -\xi^{2*} \\ \xi^{1*} \end{pmatrix}$ transforms like $\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$

which is important for antiparticle states.

Furthermore, suppose we had done this transformation with $\mathbb{B} = -\mathbb{A}$. This would have led to the same $\mathbb{H}' = \mathbb{B}\mathbb{H}\mathbb{B}^{-1} \rightarrow$ <u>same</u> $\mathbb{R}$.

2 A-like SU(2) transformations give same $\mathbb{R}$.
2:1 relationship between SU(2) and SO(3), which share the same Lie Algebra. $\rightarrow$ homomorphic.

$\longrightarrow$ go to 92 $\longleftarrow$

Pick a <u>particular</u> $\mathbb{A} = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix}$ $\quad a = e^{i\alpha/2}$, $b = 0$

Then,

$$\mathbb{H}' = \mathbb{A}\mathbb{h}\mathbb{A}^{-1}$$

$$= \begin{pmatrix} aa^* z & a^2 x - i a^2 y \\ a^{*2} x - i a^{*2} y & -aa^* z \end{pmatrix}$$

so, we see $\quad z' = |a|^2 z$

$$x' - iy' = a^2 x - i a^2 y$$
$$x' + iy' = a^{*2} x + i a^{*2} y$$

Thus, for this choice of $\mathbb{A}$ $\quad z' = e^{i\alpha} e^{-i\alpha} z = z$.

$$x' = \tfrac{1}{2}\left[x\left(e^{-i\alpha} + e^{i\alpha}\right) + iy\left(e^{i\alpha} - e^{-i\alpha}\right)\right]$$
$$x' = x\cos\alpha - y\sin\alpha \qquad !!$$
$$y' = x\sin\alpha + y\cos\alpha$$

This particular $\mathbb{A}$ does the same thing on a complex 2d vector, ξ, as

$$\mathbb{R}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & \\ 0 & 0 & 1 \end{pmatrix}$$ has on a 3d, real coordinate vector.

Just as we can build up finite rotations in $SO(3)$

$$\mathbb{R}_3(\alpha^3) = \prod_n \left[\mathbb{R}_3(n\,\delta\alpha^3)\right]^n = e^{-iJ_3\alpha^3}$$

or $$\mathbb{R}_n(\vec{\theta}) = e^{-i\vec{J}\cdot\vec{\theta}} = e^{-i\vec{J}\cdot\hat{n}\theta}$$

we have an $SU(2)$ counterpart

$$\mathbb{A} = e^{-i\vec{\sigma}\cdot\vec{\theta}/2} \Rightarrow \quad \vec{\theta} = n\theta \text{ transforms } \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$$

just like

spinor.

$$\mathbb{R}_n(\theta) \text{ transforms } \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

<u>def</u>: <u>Rank</u> is the # of operators which commutes with every generator — called Casimir Operator (J^2) — which is also the # of simultaneously diagonalizable generators

In general, for SU(n)

	SU(2)	SU(3)
The rank of the group is $n-1 \equiv \ell$	1	2
# generators for lowest dim rep $= n^2-1$	3	8
# sets of commutation relations $= \frac{1}{2}(n-1)d$	1	2