Most Important Things for You to Know about Error Analysis:

J. Linnemann

Revised August 28, 2005

Justify your uncertainty: Give a specific reason you chose δx as the uncertainty for the measurement of x. See examples in Taylor §1.5; §3.1-3.2; and §1.6, §4.1-4.6 for standard deviation and standard deviation of a mean for repeatable measurements.

Compatibility (§2.4-2.5)**: The whole point of quantitative measurement with uncertainties is to test hypotheses, and compare results.** Say you measure q, and you compare it to p (the expected value). Define the discrepancy as the difference of your result from the result expected by some hypothesis:

$\boxed{\mathbf{D = q - p = measured - expected}}$.

The best way to describe the degree of discrepancy of p and q is in terms of the number of standard deviations (the "t value") of their difference from expectations:

$\boxed{\boldsymbol{t} = \mathbf{D / \delta D}}$ where δD is the uncertainty of D (its standard deviation, for Gaussian uncertainties).

The "**two standard deviations**" rule says p and q are compatible as long as $|t| \le 2$ **.**

Typically $\delta D = \sqrt{(\delta q^2 + \delta p^2)}$; or just δq if p is well known, so δp is tiny. Best practice is to calculate *t*, then say something like "the difference is 1.6 times its uncertainty, so the measurements are compatible by the 2 standard deviation rule." If $|t| > 2$, we would call p and q statistically incompatible, or call their difference statistically significant.

If your uncertainties are Gaussian, and correctly estimated, and the assumptions (hypothesis) leading to the expected value are also correct, a $|t| > 2$ deviation would occur by chance only about 5% of the time. So large $|t|$ values suggest real disagreement from what you expected, while small $|t|$ values are compatible with your hypothesis—or at least not proven to disagree. But if you measure poorly (δD is large), your result will be compatible with most anything: not a very useful measurement.

Occasionally we use a simpler criterion compares $|D|$ with $\delta q + \delta p$ (the worst case for δD, but allowing only 1 standard deviation difference): this is just "do the error bars touch".

We are often also interested in the **fractional deviation** the measured value from what we expected, which is just $D/p = (q-p)/p$. The D(%) (**% deviation** or **% difference)** is the same thing expressed in percent. D/p or D(%) is all we can report if we don't know δD. But just because the percent difference is small, does not necessarily make it insignificantly different (statistically). That's what the t criterion is for.

Know the Uncertainty Calculation Formulae (§3.3-3.7; 3.11) **on inside covers of Taylor, and how/when to use them**. Some hints:

For $q = x \pm y$, x, y, q, dx, dy, and dq all must have the **same units** (e.g. to add q + dq: error bars)
The fraction uncertainties $\delta q/q$, $\delta x/x$, $\delta y/y$ all have NO UNITS (can write as a fraction, or as %, but watch the factor of 100!)

But to get δq, **don't forget to multiply q × (δq/q)**

How to check your calculations to see if they make sense:

$q = x + y$	always must have:	$\delta q > \max(\delta x, \delta y)$
$q = x*y$ or x/y	always must have	$\delta q/q > \max(\delta x/x, \delta y/y)$

Independent measurement: no relationship in the *imperfections* between the measurements; e.g. 2 students measure the same distance each with a different, but good, ruler. A measurement dominated by a systematic error (same shrunken ruler used by both students) would produce results that aren't independent. See Chapter 4; needed to apply Chapter 3 formulas.

Random: you expect to get slightly different values each time you measure it: due to reading uncertainties, varying judgments, uncontrollable factors, or inherent properties of the measurement.

For examples, see next page.

Standard Deviation and Standard Deviation of the Mean
The standard deviation (σ) is a measure of the uncertainty of any single measurement. The standard deviation of the mean, $\sigma_m = \sigma/\sqrt{N}$, is a measure of the uncertainty of an average of N such measurements. Clearly, the average is better known than a single measurement.

Example of q = x/y Uncertainty Calculation: $x = 10$ $\delta x = .1$ $y = 2.7$ $\delta y = .2$ so $q = 3.7$

Often easiest to do in terms of %, especially since really *need uncertainties to only 1 significant figure*

$\delta q/q = \sqrt{(1\% + 8\%)} \approx 8\%$ so $\delta q \approx .08\times q \approx .3$ (notice 8% → .08, the factor of 100)

Whip out your calculator now: Let's try $r=10$ and $\delta r = .1$, so what's the fractional error for r?

$\delta r/r = 1\%$ Now say $q= r^2$ then what's $\delta q/q = ?$

From Eq 3.23, 3.26:

$\delta q / q = (|dq/dr|\ \delta r) / r^2 = 2\ \delta r / r = 2\%$

For comparison, calculate directly (the most general way, rather than the Chapter 3 formulas, which rely on first derivative approximations):

$(q + \delta q)/q = (r+\delta r)^2 / r^2 = 102.01/100 = 1.0201 = (q + \delta q) / q$, so $\delta q/q = 2.01\%$ (same as $\delta q \to 0$)

A More Complicated Example Calculation (See Step by Step: see Taylor Chapter 3.8)

$q = x^2 y + z^{1/3}$ $x = 10 \pm .1$ $y = 20 \pm .2$ $z = 10000 \pm 1800$

$\delta x/x = 1\%$ $\delta y/y = 1\%$ $\delta z/z = 18\%$

let $w = z^{1/3} = 15.8$ $x^2y = 2000$ and $q = 2015.8$

Let's start with the product term : x^2y

$\delta (x^2y) / (x^2y) = \sqrt{\{ (\delta x^2/x^2)^2 + (\delta y/y)^2 \}} = \sqrt{\{ (2 \times 1\%)^2 + (1\%)^2 \}} = 2.2\ \% \approx 2\%$

notice we have used $\delta x^2 / x^2 = 2\ \delta x/x$: **the 2 goes inside the parentheses!**

so $\delta x^2y = x^2y \times (\delta x^2y / x^2y) = 2000 \times (2\%) = 40$

Now $\delta w/w = 1/3\ (\delta z/z) = 1/3 \times 18\% = 6\%$, so $\delta w = 6\% \times w \approx .9$

notice that 6% is NOT rounded up to 10%, nor is .948 rounded up to 1

in each instance we keep the first significant digit, though in the middle of a long calculation, it might make sense to keep one extra digit.

Notice also that w is better known than z is, and in fact has *more* significant digits: $15.8 \pm .9$ compared to $(10.0 \pm 1.8) \times 10^3$!

Finally, since $q = x^2y + w$, $\delta q = \sqrt{\{ (40)^2 + (.9)^2 \}} \approx 40$

So $q = 2015.8 \pm 40$, or $2020 \pm 40 = (2.02 \pm .04) \times 10^3$ after significant figures.