Experiment 2

Random Error and Basic Statistics

Homework 2: Turn in at start of experiment.

Readings: Taylor chapter 4: introduction, sections 4.1, 4.6 can be read together; then read the rest of chapter 4; then read chapter 5 through section 5.2. Also the experiment will refer to the summary of results from error analysis which is linked to the syllabus.

Do Taylor problems 4.2, 4.10, 4.16, and 4.23. For problem 4.2, do the calculation laid out in table style initially so you see exactly how it works; the entries in the table you can calculate either by a spreadsheet or with your calculator. But if you use a spreadsheet, you should spot-check results with your calculator! For 4.2 and 4.10, the *checking* calculations requested can be done with either your calculator, or (easier) *Excel*—but you should really do them (the purpose is to be sure you know how, and the check is that you get the right answer). You can use either explicit formulas or the built-in functions.

1. Introduction

Contrary to the naïve expectation, the experiments in physics typically involve not only the measurements of various quantitative parameters of nature. In almost all the situations the experimentalist has also to present an argument showing how confident she is about the numeric values obtained. Among other things, this confidence in the validity of the presented numeric data strongly depends on the accuracy of the measurement procedure. As a simple example, it is impractical to measure a mass of a feather using the scale from the truck weighing station, which is hardly sensitive to the weight less than a few pounds.

Another challenge has to be met when the scientist tries to compare the results of her experiment with the data from another experiments, or with the theoretical predictions. Since the conditions of the measurement almost always vary from an experiment to an experiment, and since they are also different from the idealized situation of the theoretical model, the compared values most likely will not match each other exactly. The task is then to figure out how important the factors creating this discrepancy are. If these factors are stable (**do not change from measurement to measurement**) and noticeable, they are called **systematic errors**. If, on the contrary, these factors are variable, they are called random, or statistical, errors. An important fact is that the uncertainty due to the random errors can be reduced by increasing the number of measurements to average over the random variations.

The main purpose of this experiment is to introduce you to methods of dealing with the uncertainties of the experiment. The basic procedures to correctly estimate the uncertainty in the knowledge of the measured value (*the error of the measurement*) include:

- correct treatment of the random errors and systematic errors of the experiment (*Taylor Chapters 4 and 5*);
- rounding off the insignificant digits in the directly measured and calculated quantities (Chapter 2 and the Appendix to this lab).

2. Goals

1. Understand basic statistical measures of uncertainty.
2. Learn when standard deviation vs. standard deviation of the mean is appropriate.
3. Distinguish between systematic and random errors.
4. Learn one method for estimating the random errors.
5. Learn how one way to estimate systematic errors.
6. Test for a statistically significant difference from an expected value.

3. Preliminary discussion (10-15 minutes).

Before the lab, you are asked to read and understand the theoretical material for this lab (Exp 2 and *Taylor*). Before the experiment starts, your group needs to decide which information will be relevant to your experiment. Discuss what you will do in the lab and what preliminary knowledge is required for successful completion of each step. Warning: this lab is a bit short, and the next is a bit long. **Don't leave early: but start Exp3.**

Think hard about organizing your work in an efficient way. What measurements will you need to make? Go through your lab manual with a highlighter, then make checklist of the needed measurements. What tables or spreadsheets will you need to make to organize the calculations data? How should you use *Kgraph* to expedite your calculations and unit conversions (when necessary)? What tables will you need to summarize your analysis and conclusions from the data? This lab will have more explicit reminders about tables than future labs, but you should be thinking about this organization of data taking, data reduction, and summarization in every lab.

Questions for the preliminary discussion

You should write your own answer to each question in your lab book, but leave space to change it after discussion. If you do change your answer, say why.

3.1 Calculators and computers typically return results with as many digits as possible, including digits well beyond our measurement uncertainty. What procedure will you follow to systematically get rid of these insignificant digits?

3.2 What tables will you need to record your input data? What tables will you need to summarize your results? Scan the lab, write down your tables, then see Appendix 2.

Random Uncertainties

4.1 Introduction In this experiment we ask you to perform a simple repetitive measurement in order to investigate random and systematic errors. The idea is that we will compare the time interval from the large digital "clock" at the front of the room, with a more precise instrument (the hand timers). We want to perform a test to see if the time scale of the digital clock is correct or not—that is, whether using the large digital clock would cause systematic errors were we to use it to measure time.

The problem we have is that although the timers have very accurate time scales, we need to use rather imprecise hand-eye coordination to operate them. The systematic error of the timers is small, and guaranteed by the manufacturer. But you will have to measure the random error from your hand-eye coordination, since at the start you don't know it. We will do it by repeating measurements of the same time interval on the large digital clock, and using the variation of the measurements to calculate the random error in a single time measurement. We will then use the fact that by repeatedly measuring time intervals, we can decrease the uncertainty of our estimate of the wall clock counting rate. So we are using an understanding of random errors to measure an effect (clock count rate) which might be a systematic error if used in another experiment without correcting for it.

4.2 Time Measurements

At the front of the room is a large digital clock. Assume that it counts at a constant rate, but do not assume that the rate is one count per second. The clock will count from 1 to 20, blank out for some unspecified length of time, and then begin counting again. To measure the clock's count rate, you will be given a timer whose systematic error is less than 0.001 seconds for time intervals of ten seconds or less. However, you will be relying on hand-eye coordination, which means your measurements will have random uncertainties. Your reaction time is unavoidably variable. You may also be systematically underestimating or overestimating the total time.

Observe the clock. Write down whether you believe the clock is counting reasonably close to one count per second. Also, before doing any measurements, guess how much your reaction time would vary from one measurement to the next (this is your initial guess for your random error).

Now choose a counting interval at least 10 counts long, and time 25 of them. Write down in your notebook the initial and final clock count you use to define the interval.

One person should time while the other records the data on the data sheet belonging to the person timing. To avoid an unconscious skewing of data, the person timing should not look at the data sheet until all 25 measurements have been recorded. This is essential; otherwise, you will introduce a bias into your measuring procedure! Make a few practice runs before taking data. Avoid starting a timing interval on the first count. Use the first of three counts to develop a tempo with which to synchronize your start. Exchange places

with your partner, and time 25 more counting intervals. Each person will have a data sheet with 25 timings recorded on it. Assuming you have prepared well for the lab, you should both be able to analyze your own data. In any case, you should provide your own answers to the questions at the end.

5. Data Analysis

5.1 Plot your data first!   Then if it is wildly non-Gaussian, make another trial before sinking a lot of analysis time. Use Kgraph to make a histogram from your twenty-five measurements. The x axis should represent the time T, and the y axis should be the number of measurements which fall in the kth time bin. The data should be mainly peaked at a center value, and roughly symmetrical. You can consult with your instructor to see whether to take more data.

To analyze your data, you will calculate 3 quantities, the mean, the standard deviation, and the standard deviation of the mean value. If the formulas don't make sense to you, check Taylor chapter 4 again. The average or "mean" time per interval, $\overline{T}$ (pronounced "T bar"), is

$$\overline{T} = \frac{1}{N}\sum_{i=1}^{N} T_i$$
(1)

where the Σ stands for a summation, T_i represent the I-th single measurement of the time per interval, and N = the number of measurements. This formula directs you to add up the N values of T, and divide by N. Next you can use this value of $\overline{T}$ to compute the standard deviation, σ, of the values of T, defined as:

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(T_i - \overline{T}\right)^2}$$
(2)

Finally, the standard deviation of the mean value you arrived at is related to the standard deviation of the individual values:

$$\sigma_m = \frac{\sigma}{\sqrt{N}}$$
(3)

The standard deviation of the mean σ_m, is the best estimate of the uncertainty in the measurement of the mean. Note that, unlike the standard deviation, this uncertainty can be made arbitrarily small by taking a sufficiently large number of measurements.

5.2 To demonstrate that you understand them, write out the calculations of these three quantities explicitly (say with *Excel* but not using *Excel* statistical formulas) for the first 3 measurements (N=3), as if these were the only data you had taken. You may then use the computer for the full 25 values. Find which item from Kgraph Functions | Statistics does this calculation for you (How could you check that you are looking at the right entry? *Hint*: try N=3 first before N=25). You will need the values of σ and σ_m in what follows.

5.3 Now **predict** how $\overline{T}$, σ and σ_m vary with N, the number of measurements involved in their calculation. Then use Kgraph to calculate them for the first N=3, 5, 10, and 25 (all)

of your measurements, again imagining that you had only the first 3, 5, 10, or 25 data points. Then comment whether on your predictions for the changes as N gets larger are approximately correct. In particular, why do σ and σ_m behave differently?

5.4 Compare your N=25 value of σ with your previous guess of the variability of your reaction time.

5.5 Now use Kgraph to make your final histogram from your twenty-five measurements. Adjust settings so the bin width is about w = 0.4σ Show the calculation in your notebook. How can you find the bin width Kgraph is using?

5.6 Clearly mark the points of $\overline{T}$ and $\overline{T} \pm \sigma$ for your measurements. These quantities can be shown to be the best estimate of your measurement.   The region included in the range $\pm\sigma$ should contain about 68% of your data points if your errors are random and consequently the distribution of your measurements is normal or Gaussian (Taylor, chapter 5). What fraction of your data lies within this range?

5.7 Extra Credit: (Taylor Chapter 5.2 – 5.3) Draw on your histogram an appropriate Gaussian distribution given by a curve (function) of the form

$$g(T) = Ae^{-\frac{1}{2}\left(\frac{T-\overline{T}}{\sigma}\right)^2} \tag{4}$$

To do this, you need values for the constants in the function g(T). For $\overline{T}$ and σ, use your best estimates for the mean and standard deviation of your time measurements. Choose A to match your histogram. *Hint*: what is the value of the function g(T) at T=$\overline{T}$?

Calculate g(T) at five points (*Hint*: *Kgraph* and *Excel* both use *exp* for the exponential function.). Draw the points by hand on your histogram plot. Then connect these points with a smooth curve, which should resemble the Gaussian curves in Taylor. With a finite number of measurements such as 25, your histogram may not resemble the expected "bell" shape curve to a great degree.

6. Drawing Conclusions from the Timing Measurements

6.1 Calculate the time interval per clock tick (How is this related to $\overline{T}$?). Does it appear that the clock tick is 1.0 seconds, as we assumed at the beginning? By how many % does your time differ from 1.0 seconds? What is the uncertainty in your estimate?

6.2 Next we will check to see is there a (statistically) significant discrepancy in the time measured by the large clock? We want to check the hypothesis that the large digital clock is running correctly. Statistical significance is tested for not by just looking at the size of the difference and saying "that seems small" or "looks big to me". Rather we compare

the relative size of the difference with the uncertainty of our measurement. If the difference isn't substantially larger than our uncertainty, then we say that our statistical analysis left us with information insufficient to reject our original hypothesis, and we say that the differences are not statistically significant. We make the test quantitative by calculating the size of the discrepancy from expectations in units of the uncertainty of that difference. The discrepancy we want is that between the average time counted, $\overline{T}$, with its expected value $T_{exp} = (1 \text{ count/s}) \times$ (the number of counts in your interval). Following the **uncertainties summary** (linked to the syllabus), we use $D = \overline{T} - T_{exp}$. We use σ_m for δD, since that is our uncertainty in how well we know $\overline{T}$, and there is no uncertainty in our prediction, T_{exp} . Then

$$ t = \frac{\left|\overline{T} - T_{exp}\right|}{\sigma_m} \tag{5} $$

This expression is the same as Eq. 5.67 on p. 150 of your text, and one of the boxed equations in the summary on uncertainties. Here, in accord with standard statistical notation, t has the meaning of the number of the standard deviations of the mean needed to cover the difference between the mean and expected times. **In this equation, t is not a time**: it has no units since both the numerator and denominator are in seconds. The statistic t is handy because we can make such a comparison for any measurement: the numerator D and denominator δD always have the same units, so t is always means "number of standard deviations", a pure number on the same (dimensionless) scale, no matter what quantity was originally being measured. Following the handout, and common statistical practice, we will say the deviation is **statistically significant if |t|>2**, that is the discrepancy is more than twice as big as our uncertainty.

As you will see later, this should happen by chance only about 5% of the time. Notice also if we make a sloppy measurement with a huge uncertainty δD, it will be hard to ever learn anything: it would be rare to have a large enough discrepancy to find a statistically significant difference from our starting assumption.

Based on this calculation, is $\overline{T}$ compatible with T_{exp} ? That is, is the discrepancy (statistically) significant?

6.3 Why is σ_m used instead of σ in the formula (5) above?

6.4 Suppose you used the timer at the front of the room for timing measurements. Would doing so cause a systematic error in measurements of time intervals?

6.5 Suppose the difference *was* statistically significant and you needed to use the large digital clock as a timer on another experiment. Based on your data, can you correct for its systematic error? Explain how you would give the best estimate (in seconds) of a time interval of 120 counts. What uncertainty would you report for that time interval?

6.6 Suppose you had recorded only your first 5 measurements. What would you have concluded about the existence of a significant discrepancy? Were the remaining 20 measurements necessary in your opinion? Give a quantitative justification.

6.7 Suppose you measured with the hand timer a different counter with a known period, and found your measured period was (statistically significantly) too low. What kind of flaw in your measurement procedures might have caused this bias?

6.8 What was the muddiest point of this experiment? Where, specifically, should the write-up be improved?

Appendix 1. Significant Figures

In the calculations, it is always important to distinguish significant from insignificant figures in the final presented numbers. The rules in Taylor section 2.2 describe things rather well. We add the following comments.

1. Since the error of the measurement is only an approximate <u>estimate</u> of the uncertainty of the measurement, we do not need to keep more than one or two largest digits in it. The smallest digit in the mean value should be of the same order as the smallest significant digit of the uncertainty. Examples:

Incorrect	Correct
517.436 m $\pm$ 12.34 cm (avoid mixed units)	517.4 $\pm$ 0.1 m or 517.44 $\pm$ 0.12 m
24.3441364 $\pm$ 0.002 m/s	24.344 $\pm$ 0.002 m/s
12385 s $\pm$ 241 s	12400 $\pm$ 200 s or 12390 $\pm$ 240 s or $(1.239 \pm .024) \times 10^4$ s or $(1.24 \pm .02) \times 10^4$ s

2. When adding or subtracting two numbers, the result should have the same number of the significant digits <u>after the decimal point</u> as the least precise summand. Example:
$$517.45 + 34.7824 = 552.23$$
3. When multiplying or dividing, the result should have the same <u>total</u> number of the significant digits as the least precise multiplier. Example:
$$1234.3 \times 23.45 \approx 28940$$
4. For other operations (raising to power, square root, exponent, sine) the rule is similar to the one for the multiplication and division: you should keep as many significant digits in the final result as you had in the input. Example:
$$\sqrt{3.567} \approx 1.889$$

These rules apply for presentation of final results. During multi-step calculations, keep one or 2 extra decimal places to avoid unintentional loss of significance.

Appendix 2: Summary Table for Experiment 2:

N=	all times in seconds		
	mean	std dev	std dev mean
3			
5			
10			
25			