Experiment 5

The Simple Pendulum

Reading:

Read Taylor chapter 5. (You can skip section 5.6.IV if you aren't comfortable with partial derivatives; for a simpler approach to the material of section 5.7, see below).
Halliday, Resnick, Walker Ch 8, 14 as needed (pendulum, and definitions of period, amplitude)

Taylor Section 5.7 without partial derivatives:

We want to know the uncertainty of the mean value, $m = (x_1 + x_2 + \ldots + x_N)/N$. Now N is a constant, so I can start by determining the uncertainty of the sum $S = N\,m = (x_1 + x_2 + \ldots + x_N)$. But by the arguments of 5.6.III (and Eq. 3.16), the uncertainty of S is just given by:

$$\sigma_S = \sqrt{\{\sigma_x^2 + \sigma_x^2 + \ldots + \sigma_x^2\}} = \sqrt{\{N\sigma_x^2\}} = \sigma_x \sqrt{N}$$

since the standard deviation of each of the x_i is just σ_x

But $m = S/N$, and N is a constant, so by eq. 3.9, $\sigma_m = \sigma_S / N$, which gives our final result

$$\sigma_m = (\sigma_x \sqrt{N}) / N \text{ or } \quad \boldsymbol{\sigma_m = \sigma_x / \sqrt{N}}$$

Homework 6: turn in as part of your preparation for this the first week of this experiment.

For the first two problems, don't just write down the answers, but explain how you got them.
1. Some statisticians speak of a "68-95 rule". What do you suppose this rule is? What is the "68-95-99.7 rule"?
2. We often say that the probability of a measurement in the range $X \pm 2\,\sigma$ is about 95%. What is this probability to four significant figures? What range $X \pm t\,\sigma$ actually has probability equal to 95.00%? (Give t to three significant figures. The probabilities can be found in Taylor Appendix A.)
3. Define and briefly explain the following terms: a) amplitude b) period c) frequency and d) angular frequency.
4. Write down the general equation relating period and frequency
5. Write down the general equation relating frequency and angular frequency.

Homework 7: Turn in at start of 2nd week of experiment.
Don't just write down the answers, but explain how you got them.
1. According to a proposed theory, the quantity x should have the value x_{th}. Having measured x in the usual form $x_{meas} \pm \sigma$ (where σ is the appropriate SD), we would say that the discrepancy between x_{meas} and x_{th} is t standard deviations, where $t = | x_{meas} - x_{th} | / \sigma$. How large must t be for us to say the discrepancy is significant at the 5% level? At the 2% level? At the 1% level?
2. A student measures g, the acceleration of gravity, repeatedly and carefully gets a final answer of 9.5 m/s2 with a standard deviation of 0.1. If his measurements were normally distributed with a center at the accepted value of 9.8 and with a width of 0.1, what would the probability of his getting an answer that differs from 9.8 by as much (or more than) his? Assuming he made no actual mistakes, do you think his experiment may have suffered from some undetected systematic errors?
3. For a given initial angle Θ, calculate the h, the maximum height of the bob above equilibrium used in Eqs 15-16 in the lab writeup below.

4. Extra Credit: **Weighted averaging (How grades are calculated, for example)**:
Taylor chapter Eq. 7.10 gives the general equation for weighted averaging. A special case is
Taylor Eq 5.7 (and 5.6), which apply this definition to histograms, with the definition of the
weights obeying Eq 5.8. A convenient way to calculate weighted averages is to make an Excel
spreadsheet like the following, which uses the data histogram of Taylor table 5.1 and figure 5.1.

value Xi	weight Wi	product Xi * Wi	value Xi	weight Fi	product Xi * Fi
23	1	23	23	0.1	2.3
24	3	72	24	0.3	7.2
25	2	50	25	0.2	5
26	3	78	26	0.3	7.8
27	0	0	27	0	0
28	1	28	28	0.1	2.8
					Eq. 5.7 (or 7.10) weighted average:
sum	10	251	sum	1	25.1
		Eq. 7.10 weighted average:			
		25.1			
Mean Xi:			Mean Xi:		
25.5			25.5		

The first calculation uses formula 7.10 directly. The second uses weights defined by Eq 5.6,
which are forced to sum to 1.0. Since either gives the same weighted average, there is no
intrinsic requirement that the weights sum to 1.0, provided you use Eq. 7.10 .

Calculate appropriate weighted averages for the following situations. Give also the
(unweighted) mean for comparison. This can be calculated by the Average spreadsheet function.
You only need a table like the left part of the spreadsheet above.

4a) Grade Point Average: You earn for 2.5 for Physics 183 (4 credits); 3.5 for physics 191 (1
credit); 3.0 for Chemistry 251 (3 credits). The course credits are the weights.

4b) Lab average: your scores on the first 3 labs are 50%, 70%, and 90%. These labs are
weighted by weeks, with weights of ½ week, 1 week, and 2 weeks.

4c) Lab grade: Your grade will be determined by quizzes (10%), homework (10%), labs (65%)
and a practical exam (15%). Your scores are 60%, 70%, 70%, and 90% respectively.

5. Extra Credit: Verify by direct substitution that Eq. 7 below is a solution of Eq. 6.

Goals

1. Measure g with the simple pendulum
2. Improve measurement accuracy by averaging
3. Study the amplitude and mass dependence of the period of a pendulum
4. Study energy conservation
5. Examine the propagation of error in derived physical quantities

1. Theoretical Introduction
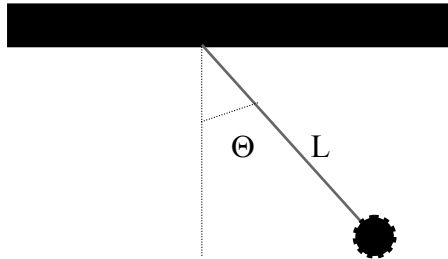
1.1 The period of the pendulum

The simple pendulum, shown above, consists of a mass m (the "bob") suspended from a pivot by (ideally) a massless string. We will idealize the bob as a point mass located at the center of mass of the bob. The distance from the point of pivot to the center of mass of the ball is designated by L in figure. When the ball is displaced from its resting positions the string makes an angle Θ with the vertical. The component of the gravitational force in the tangential direction acts to restore it to its equilibrium position. Calling x the tangential coordinate ($x = L\Theta$, the arc length, with $x=0$ at $\Theta=0$), the restoring force is:

$$F_x = -mg\sin\Theta \qquad\qquad (1)$$

The tension of the string T, in the direction toward the point of suspension, is equal in magnitude and opposite in direction to the component of the gravitational force acting in that direction. The mass is accelerated only in tangential direction perpendicular to the string. Using Newton's Second Law, F= ma, the relation between the tangential displacement x and the corresponding change in angle Θ is given by:

$$F_x = ma_x = m\frac{d^2x}{dt^2} = m\frac{d^2}{dt^2}(L\Theta) = mL\frac{d^2\Theta}{dt^2} \qquad\qquad (2)$$

Inserting the expression for the restoring force, our equation of motion becomes:

$$mL\frac{d^2\Theta}{dt^2} = -mg\sin\Theta \qquad\qquad (3)$$

or

$$\frac{d^2\Theta}{dt^2} + \frac{g}{L}\sin\Theta = 0 \qquad\qquad (4)$$

This turns out to be hard to solve, but we can simplify it by using the fact that for small angles Θ, and Θ expressed in radians, we may expand $\sin\Theta$ as follows :

$$\sin \Theta \approx \Theta - \frac{\Theta^3}{6} + \frac{\Theta^5}{120} + \text{higher order terms} \qquad (5)$$

Then, if we assume Θ to be small and keep only the first term, Eq 4 becomes:

$$\frac{d^2\Theta}{dt^2} + \frac{g}{L}\Theta = 0 \qquad (6)$$

This is much easier to solve analytically, and the solution to differential equation (6) is:

$$\Theta = \Theta_o \sin(t\sqrt{\frac{g}{L}}) \qquad (7)$$

Because the sine function repeats itself whenever its argument changes by 2π, we can find T, the time for one period, by setting the argument of the sine to 2π:

$$T\sqrt{\frac{g}{L}} = 2\pi \qquad (8)$$

or

$$T = 2\pi\sqrt{\frac{L}{g}} \qquad (9)$$

Now we can solve for g in terms of two quantities we can measure. Or, we can find a relationship between measured quantities such that g is related to the slope of a plot when we vary the values:

$$g = 4\pi^2 \frac{L}{T^2} \quad or \quad L = \frac{1}{4\pi^2} g T^2 \qquad (10)$$

Thus as long as the small angle approximation of Eq. 5 is valid, the period is independent of the amplitude, Θ_o and mass, m. Measurement of the period T , and the length L , permit a determination of the gravitational constant, g. If Θ_o is not small enough, Eq. 6 will not be valid and the period will depend on Θ_o and will actually increase with the amplitude (see Appendix B).

1.2 Energy analysis of the pendulum

For a pendulum swinging back and forth, the mechanical energy, E, shifts between kinetic and potential energy, but remains constant:

$$E = K + U \qquad (11)$$

$$U = mgy \qquad (12)$$

$$K = \frac{1}{2}mv^2 \qquad (13)$$

Here y is vertical displacement from equilibrium, and v is velocity of the bob. When the bob is at the maximum amplitude, $x = x_m$, and $y = h$ (the maximum vertical displacement). At this point, $v = 0$: there is no kinetic energy, so all the energy is potential energy.   The bob has greatest speed at its lowest point, hence all the energy is kinetic, and $U = 0$.

Conservation of mechanical energy for these two instants can be expressed as:

$$K_0 + U_0 = K_m + U_m \tag{14}$$

where subscript 0 stands for values evaluated at the equilibrium position ($x = 0$, $y=0$) and the subscript m stands for values at highest point of the oscillation ($x = x_{max}$, $y=h$). Then we can evaluate each term and find

$$\frac{1}{2}mv_0{}^2 + 0 = 0 + mgh \tag{15}$$

This equation relates the maximum velocity (at x=0) to the maximum height and the value of g. It also allows us to express the total kinetic energy to the height under the assumption of energy conservation. Curiously, the maximum velocity is achieved at the equilibrium position!

We can also solve (15) for $v_0{}^2$ and obtain:

$$v_o{}^2 = \left(\frac{\Delta x}{\Delta t}\right)^2 = 2gh \tag{16}$$

This expression will be useful when we study energy conservation.

Experimental Procedure

2. Preliminaries

A bob is suspended from a pivot by a string. A protractor is placed below the pivot which allows us to set the pendulum oscillation at different angles (amplitude). We can measure the period of the pendulum using:

       1. Manual timing with a digital clock.
       2. Automatic timing with a photogate timer.

You should then be able to compute g from the period of the pendulum and the length of the string.

2.1 Measure the length L between the pivot of the pendulum and the center of mass of the bob as accurately as possible. You may need several measurements, or measurement strategies, to find L. **Assess the role of statistical and systematic uncertainties in your value for L.** *Hint:* watch the support point as the pendulum oscillates.

2.2 Questions for preliminary discussion

2.2.1 Draw a diagram of forces acting on the bob.
2.2.2 What component of force causes oscillation?
2.2.3 How should the period of the pendulum depend on the mass?
2.2.4 How should the period of the pendulum depend on the amplitude?
2.2.5 What is the criterion for a small-amplitude oscillation?
2.2.6 Would the measurements be most accurate with a long or a short string?
2.2.7 Discuss sources of systematic and random errors in this experiment.
2.2.8 What tables will you need to organize your measurements and the uncertainty calculations?

3. Manual measurements of T:

It is most accurate to begin timing the swing of the pendulum at its lowest point because then the ball moves most quickly and takes the least time to pass by: the accuracy of determining the low

point is not limited by the motion of the pendulum. The amplitude of the swing should be large in order to maximize the speed of the pendulum at that point. On the other hand, Θ must be kept small enough that the approximation $\sin \Theta \approx \Theta$ remains valid. As a compromise, take the initial amplitude to be about 0.1 radians ($\sim 6^{o}$).

3.1 Using the timer, measure 25 complete cycles From this measurement of the time for 25 cycles, calculate the period. Now repeat the period measurement **five times**. *Hint:* So you can end when your count gets to 25 cycles, count "zero" when you start the timer. One complete cycle is from when it's going right at the lowest point of the swing until when it returns to the bottom twice and is again going right.

3.2 From these five measurements of the period, calculate the mean period and the standard deviation of the mean period using Kgraph or Excel. Do not round off too early in your calculations lest you lose accuracy in your final result. Using Eq. 10, calculate *g.* Also calculate the uncertainty of your value of *g.*

4. Automatic measurement of T (for various masses and lengths)

Set the photogate on PENDulum position. Practice timing the period with using the photogate a few times. Figure out how the gate works by moving the bob through the gate *slowly, by hand.* How many periods does the photo-gate measure? (Write it in your lab book!)

4.1. From three measurements of the period with the photo-gate, again calculate the mean, and, the standard deviation of the mean. Calculate g and its uncertainty.

4.2. Find the period for bobs of other masses. Are your results consistent with Eq 10?

4.3. Now measure the period with the length of the string reduced to approximately L/2, L/3, L/4, and L/5. Make a plot of T^2 vs. *L* . Figure out how g is related to the slope of this plot and find and *g* and its uncertainty from this method. (For Kgraph to calculate uncertainties for you, you'll need error estimates for L).

4.4. An oscillating solid rod with uniform cross section also forms a pendulum. If the rod is suspended from one end, its period is given by $T = 2\pi \sqrt{\dfrac{2L}{3g}}$. (This result is derived in Appendix A.) Find the period of the rod as in (4.1) and compare it to the prediction (difference in % is sufficient).

5. Amplitude dependence of the period

If the amplitude of oscillation of a pendulum is not sufficiently small, its period will depend on amplitude. Thus Eq. 9 will not be valid. See Appendix B for a brief discussion.

For this part, remove the rod and go back to the string and bob. Hint: the largest bob may not be appropriate for this measurement (why?). Use the photogate timer to measure the period of the pendulum for a series of starting angles. Begin with 30° (about 0.5 radians) maximum and repeat for approximately 25°, 20°, 15°, 10°, and 6°. Now we have data to test Eq. B3, which has the form

$$T = T(\Theta) = T_o\,[1 + A\,\Theta^2 + \text{higher terms}],$$

where To is the small amplitude period defined in Eq. 9, and A = 1/16 **(provided Θ has been converted to radians)**. So a nice way of testing the equation is to calculate the ratio $T(\Theta)$ / To as a function of Θ^2, where $T(\Theta)$ is the period you measured with a given initial angle. If Eq B3 holds, and the higher terms are negligible, the plot should be close to linear. Perform a linear least squares fit of the data to T(Θ)/To vs. Θ^2 . Compare your slope with the theoretical value of 1/16. It is not necessary to find the uncertainty of the slope: just say by how many percent your coefficient differs from the expected one, and whether the data follow the expected trend.

6. Conservation of Energy

Now we will test the idea of conservation of energy by measuring the velocity of the bob (kinetic energy) as a function of its release height (potential energy). Measure the diameter of the bob with maximum accuracy. Set the photogate in *Gate* position. Determine the vertical displacement from equilibrium h for the angles $\Theta = 30°$ to 5° in 5° intervals. When the bob passes the equilibrium point, the photogate timer measures the time interval over which the bob interrupts the light. From the time intervals, find v_0 and plot v_0^2 vs. h (testing Eq. 16). Perform a linear least-squares fit to the data to obtain g. It is not necessary to do an extended uncertainty analysis here, either—just discuss whether the data follows the expected trend and calculate the value of *g* you obtain from the slope (by what % does it deviate from the accepted value?).

7. Damped Pendulum

Next we consider the damped pendulum. In any real pendulum, frictional losses decrease the energy of the pendulum as time goes on. We now concentrate on this aspect of the pendulum by measuring the peak velocity of the pendulum as its swings slow down. This will allow you to find the functional form of the energy decay. You don't need error bars for this measurement.

Choose a string length of about 20 cm and set the photogate timer switches to GATE with MEM *on*. Take 10 successive measurements of the bob velocity, at 30 s intervals. Use the manual timer to say when to make a new velocity measurement. At each interval, determine the velocity by reading the photogate timer to measure Δt and thus the velocity. Also record the time since the last measurement. Your final kinetic energy should be < 5% of the initial value. If not, choose a different interval for your measurements.

Calculate *E* (recall from Eq 15 that the potential energy is 0 at the bottom of the swing) and plot the kinetic energy vs. total elapsed time at which you made the velocity measurement. Explain the graph in your report: what is it telling you? **Extra credit:** Can you transform your plot variables so that a straight line results? Can you define a characteristic time for the energy decay?

8. Phase Space Portrait (Extra Credit; no data required)

When the bob is released from an angle Θ , the pendulum oscillates between $+\Theta$ and $-\Theta$. In one cycle, the bob passes each point twice. The momentum *p* at these two passages is first positive, and then negative. "Phase space" considers motion with both position and momentum as coordinates, so it considers these two passages as two distinct points (θ, p) and (θ, -p); in phase space, the cycle is not complete until the object returns to the same phase space point.

Make a *phase space* plot describing the motion of a pendulum. Your phase space plot should have orthogonal axes for *p* and Θ . For example, assume that $\Theta = 30°$ is your initial angle. Then

when $\Theta = 30°$, $U=mgh$ and $p = 0$. When $\Theta = 0°$, $p= -mv$ (left direction is negative; we've just assumed a symbolic value for p). At $\Theta = -30°$, $p = 0$ again; and on returning to $\Theta = 0°$, $p = +mv$ (going right). Connect these four points with smooth line, forming an ellipse. Repeat for smaller initial angles. Now, imagine that the pendulum's amplitude is continuously decreasing to zero. Sketch the diagram for this situation. For a pendulum that is being damped, the diagram encompasses its complete dynamics, from start to finish. Explain this in your report.

9. Analysis of Results

9.0 Make a summary table listing all your methods of measuring g, the values and (if applicable), the uncertainty (absolutely, and in %), the percent difference from the accepted value of g (9.804 m/s^2), and where possible, the t value of the discrepancy between your measured value and the expected one.

Address the following questions in your report:
9.1. What is the point of measuring 125 cycles (25 × 5) in Part A? Would it have been as accurate to measure one cycle 125 separate times? Why?
9.2. Which value of *g* is more accurate, the one obtained by hand-timing or obtained with the photogate? Which method is faster to use?
9.3. Assuming that your uncertainties are random, how many hand-timing measurements should be done to make the two sets of measurements equally precise? *Hint*: how does the uncertainty depend on the number of measurements done?
9.4. Which quantity, T or L, makes the larger contribution to the fractional uncertainty in *g*? Does this suggest a way to improve the experiment?
9.5. Compare your most precise value of *g* with 9.804 m/s^2. Do you have a significant discrepancy? Discuss this quantitatively from a statistical point of view.
 9.6 Which experiment measured *g* to better fractional uncertainty, the Pendulum, or Free Fall?
9.7. What the largest source of error in your experimental verification of $v_0^2 = 2gh$?
9.8. For which parts of the experiment did your results accord with your expectations? Which did not? Why?
9.9 The kinetic energy of the pendulum decreases during the 25 cycles. Given your results, what would you estimate as a maximum systematic error for this effect?
9.10 Was the value you used for the length of the rod corrected for the fact that the pivot wasn't at the end? Not correcting would cause a systematic error. With no correction, would your measured period be higher, or lower, than the value predicted using the full length of the rod?
 9.11 **Extra Credit** Suppose that the width of the photogate beam of light is finite, of width w. For a bob of radius r, calculate the effect (if any) on the value of g obtained, and on the value of kinetic energy compared with potential energy. *Hint:* draw a diagram.
9.12 **Extra Credit**   Suppose the plane of oscillation of the bob is not perpendicular to the light beam of the photogate. Find an expression giving the effect (if any) on the measurement of the pendulum period, and of its velocity.

Appendix A. The Solid Rod Pendulum

For a physical pendulum like a rod pivoting at one end, a restoring torque τ reduces the angle when the pendulum is displaced from its equilibrium,
$$\tau = (mg \sin \Theta) (d) , \qquad\qquad\qquad\qquad\qquad (A1)$$

where $mg \sin \Theta$ is the tangential component of the force and d is the distance from the pivot to the center of mass.

For small angles Eq. A1 becomes:

$$\tau \approx -(mgd)\Theta \tag{A2}$$

which is the angular form of the Hooke's law. Writing the differential form of the torque and setting it equal to Eq. B2, we obtain the equation of motion:

$$\tau = rF = rm\frac{d^2(r\Theta)}{dt^2} = mr^2\frac{d^2\Theta}{dt^2} = -(mgd)\Theta \tag{A3}$$

or

$$\frac{d^2\Theta}{dt^2} = -(\frac{mgd}{I})\Theta \tag{A4}$$

where $I = \sum mr^2$ is the moment of inertia. The solution to (A4) is:

$$\Theta = \Theta_o \sin\left(t\sqrt{\frac{mgd}{I}}\right) \tag{A5}$$

so that the time for one period is

$$T = 2\pi\sqrt{\frac{I}{mgd}}. \tag{A6}$$

For the rod $I = \frac{1}{3}mL^2$ and the center of mass distance d=L/2 where L is the length and m is the mass of the rod. Therefore, we find that

$$T = 2\pi\sqrt{\frac{2L}{3g}} \tag{A7}$$

Appendix B. The Finite Amplitude Pendulum

We wish to solve Eq.4 exactly:

$$\frac{d^2\Theta}{dt^2} + \frac{g}{L}\sin\Theta = 0 \tag{B1}$$

We may formally write the solution for the period, T, as an integral over the angle Θ (in radians):

$$T = 2\sqrt{\frac{L}{g}}\int_0^{\Theta_o}\left[\left(\sin\frac{\Theta}{2}\right)^2 - \left(\sin\frac{\Theta_o}{2}\right)^2\right]^{-\frac{1}{2}}d\Theta \tag{B2}$$

Integrals of this form belong to a class of elliptic integrals which do not have closed form solutions. If the angle Θ_o is sufficiently small, solutions to any desired degree of accuracy can be obtained by doing series expansions in the angles. We state the result without proof below, which is assumes the angle Θ_o is in radians:

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1}{16}\Theta_o^2 + \frac{11}{3072}\Theta_o^4 + \cdots\right) \tag{B3}$$

Note that the period *increases* as the amplitude *increases*.