
PHY481: Electromagnetism

Another image problem

Problems with specific symmetries

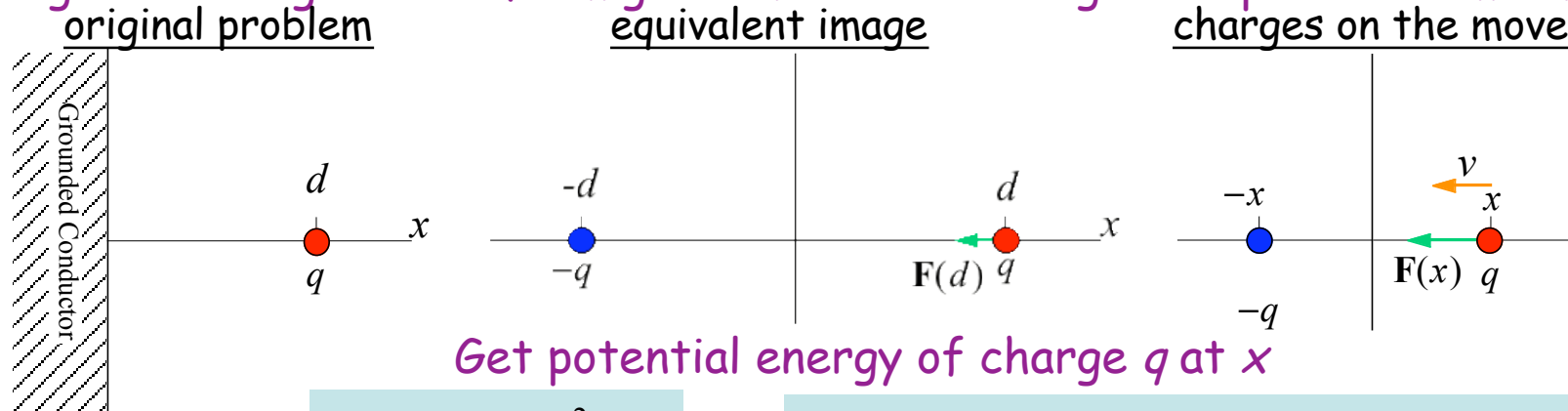
Solutions via "Separation of Variables"

1) Cartesian coordinates

Practical image problem (Lecture 18)

Charge q , mass m is released a distance d from a grounded plane.

Negative charge drawn from ground. Positive charge hits plane at time t .



Get potential energy of charge q at x

$$F(x) = \frac{-q^2}{4\pi\epsilon_0 (2x)^2} \hat{\mathbf{i}}$$

$$U(x) = -\int_{\infty}^x \mathbf{F} \cdot \hat{\mathbf{i}} dx' = \frac{q^2}{4\pi\epsilon_0} \int_{\infty}^x \frac{dx'}{(2x')^2} = \frac{-q^2}{16\pi\epsilon_0 x}$$

Note: $(2x)^2$

Change in PE from d to x

E conservation: $\Delta KE = -\Delta U(x)$

Use energy conservation

$$\Delta U(x) = U(x) - U(d) = \frac{-q^2}{16\pi\epsilon_0} \left[\frac{1}{x} - \frac{1}{d} \right]$$

$$\frac{1}{2}mv^2 = -\Delta U(x) = \frac{q^2}{16\pi\epsilon_0 d} \left[\frac{d-x}{x} \right]$$

solve for v

$$\frac{dx}{dt} = v = \frac{q}{\sqrt{8\pi\epsilon_0 md}} \sqrt{\frac{d-x}{x}}$$

integrate over t and x'

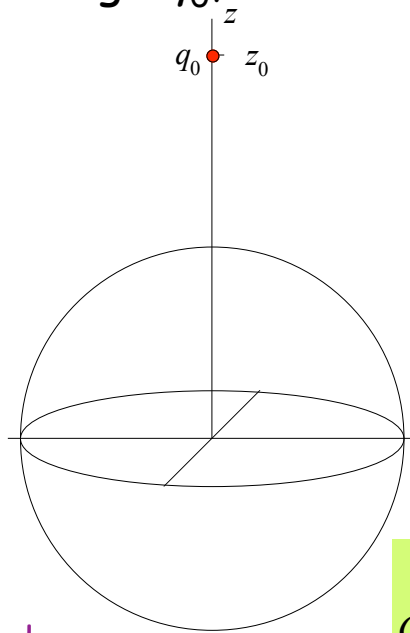
$$\int_0^t dt' = \frac{\sqrt{8\pi\epsilon_0 md}}{q} \int_d^0 \sqrt{\frac{x'}{d-x'}} dx'$$

<http://integrals...>

$$t = \frac{\sqrt{8\pi\epsilon_0 md}}{q} \frac{\pi d}{2}$$

Spherical image problem

Charge q_0 , a distance z_0 from the center of a grounded sphere. Force on q_0 ?



V must be zero at these points

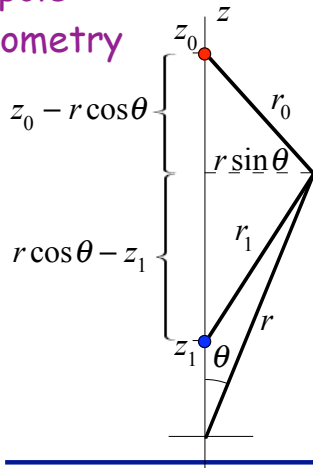
$$\frac{q_0}{z_0 - R} = \frac{-q_1}{R - z_1}$$

$$\frac{q_0}{z_0 + R} = \frac{-q_1}{z_1 + R}$$

simultaneous eq.

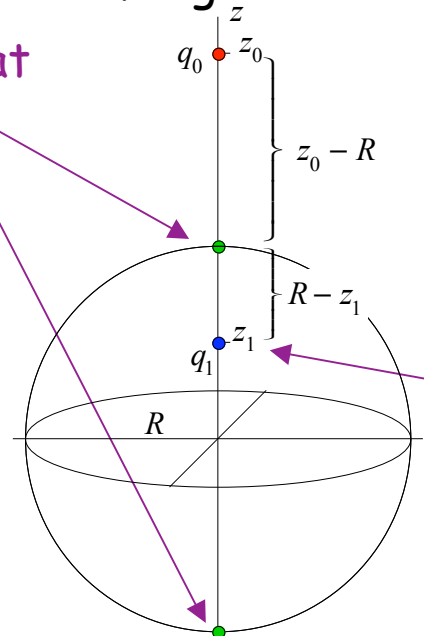
$$q_1 = -q_0 \frac{R}{z_0}; \quad z_1 = \frac{R^2}{z_0}$$

Dipole geometry



Dipole potential

$$V(r, \theta) = \frac{1}{4\pi\epsilon_0} \left[\frac{q_0}{r_0} + \frac{q_1}{r_1} \right]$$



Replace sphere with image charge q_1 , a distance z_1 from the center.

Force on q_0

$$\begin{aligned} F_z &= \frac{q_0 q_1}{4\pi\epsilon_0 (z_0 - z_1)^2} \\ &= \frac{-q_0^2 (z_0/R)}{4\pi\epsilon_0 R^2 [(z_0/R)^2 - 1]^2} \end{aligned}$$

Field & PE

$$\mathbf{E} = -\nabla V$$

$$W = U = - \int_{\infty}^{z_0} F_z dz$$

Solving Laplace's equation (1)

1) Grounded conductor and an external charge or dipole:

Potential solved by the Method of Images

Rectangular, Spherical, and Cylindrical symmetry

2) Potential between conductors: capacitors $Q = CV$

Parallel plate capacitors $V(z) = Az + B$

Spherical capacitors $V(r) = Ar^{-1} + B$

Cylindrical capacitors $V(r) = A \ln(r/r_0) + B$

3) Potentials for conductors in external fields

Spherical symmetry $V(r, \theta) = Ar^{-1} + B + \frac{C \cos \theta}{r^2} + Dr \cos \theta$

Cylindrical symmetry $V(r, \phi) = A \ln\left(\frac{r}{r_0}\right) + B + \frac{C \cos \phi}{r} + Dr \cos \phi$

Solving Laplace's equation (2)

In general, a solution of Laplace's equation subject to boundary conditions requires an infinite series of functions.

Fourier series

Any periodic function with period $2a$, can be expanded in a "Fourier Series" of sine and cosine functions, with n a positive integer

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{a} + b_n \sin \frac{n\pi x}{a} \right]$$

"Orthogonality" of the trigonometric functions

$$\frac{1}{a} \int_{-a}^a \cos \frac{n\pi x}{a} \cos \frac{m\pi x}{a} dx = \delta_{nm}$$

$$\frac{1}{a} \int_{-a}^a \sin \frac{n\pi x}{a} \sin \frac{m\pi x}{a} dx = \delta_{nm}$$

The coefficients a_n and b_n are determined by doing these integrals

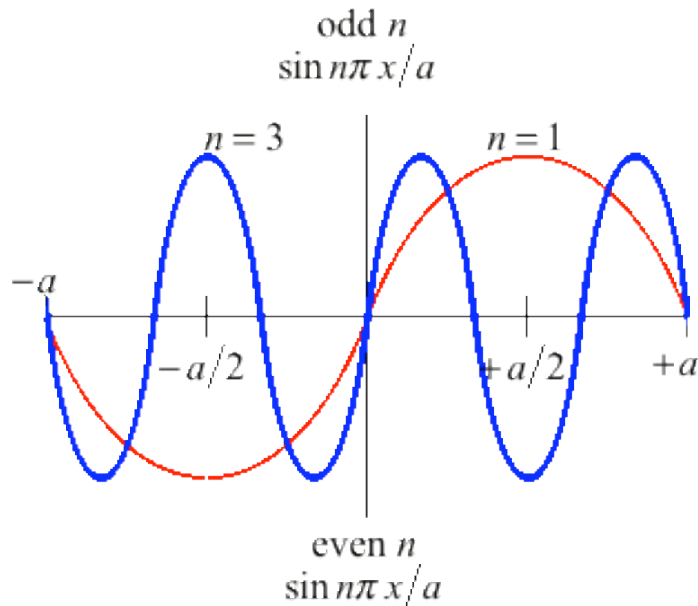
$$a_0 = \frac{1}{2a} \int_{-a}^a f(x) dx$$

$$a_m = \frac{1}{a} \int_{-a}^a f(x) \cos \frac{m\pi x}{a} dx$$

$$b_m = \frac{1}{a} \int_{-a}^a f(x) \sin \frac{m\pi x}{a} dx$$

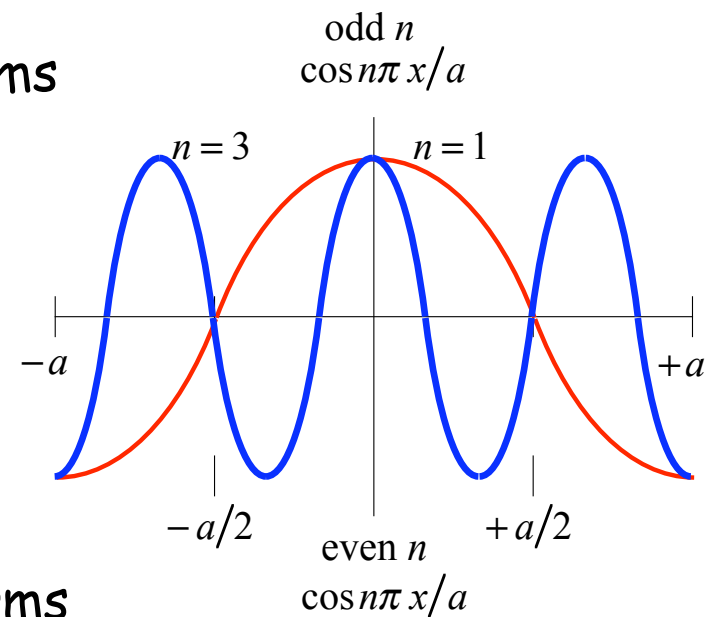
Odd or even functions

Odd functions select these



Odd n terms

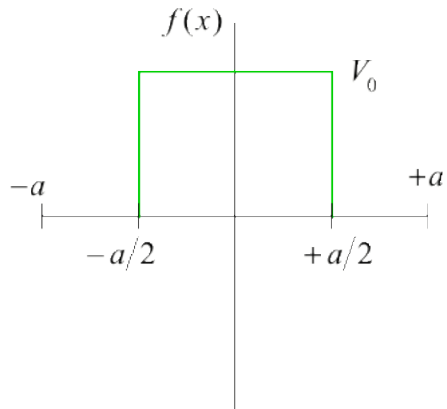
Even functions select these



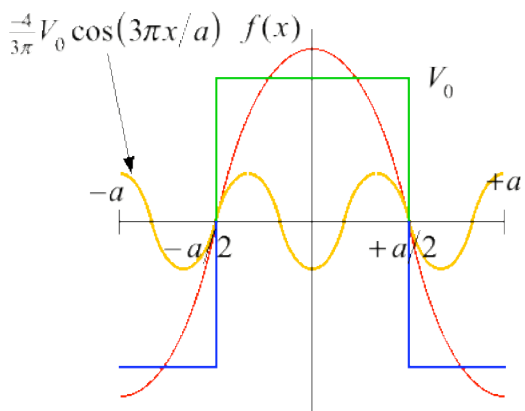
Even n terms

Qualitative expansion of $f(x) = V_0$

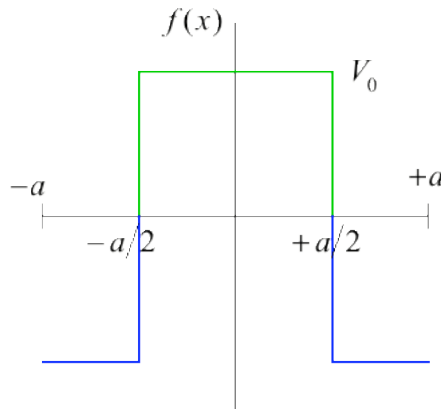
Fourier series expansion
 $f(x) = V_0 \quad (-a/2 < x < +a/2)$



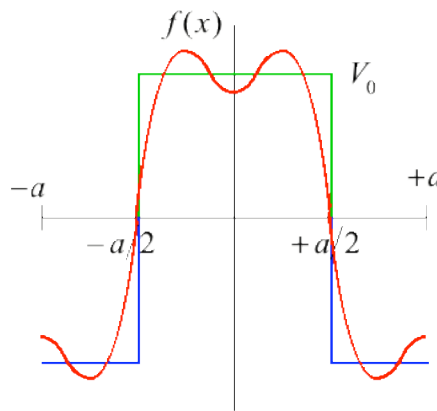
Prepare a fraction
of the next odd n cosine



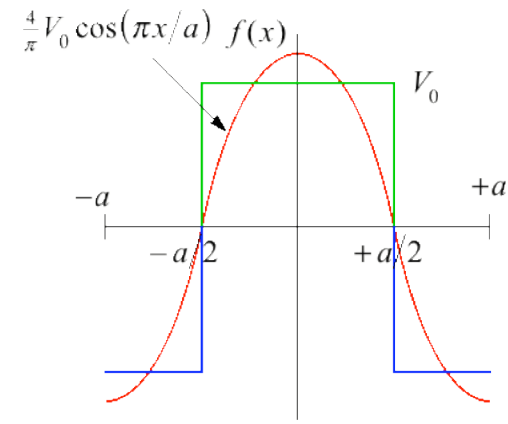
Extend $f(x)$ to make
a periodic function



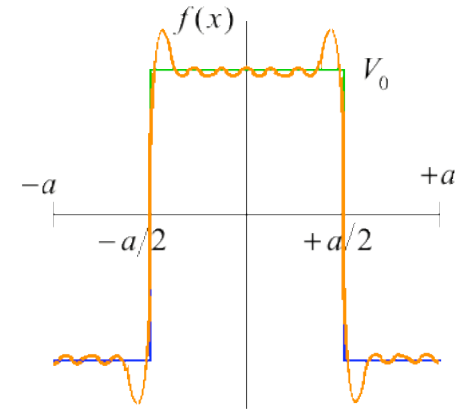
Add to the first.
Getting closer!



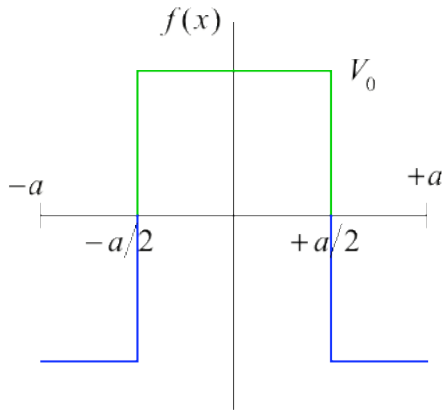
Note similarity to
cosine function



Add a few more terms
Pretty good match!



Coefficients in a Fourier Series



Find Fourier series expansion of $f(x) = V_0$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{a} + b_n \sin \frac{n\pi x}{a} \right]$$

$$a_m = \frac{1}{a} \int_{-a}^a f(x) \cos \frac{m\pi x}{a} dx$$

$$b_m = \frac{1}{a} \int_{-a}^a f(x) \sin \frac{m\pi x}{a} dx = 0$$

function $f(x)$ is even
so no sin terms

$$= \frac{1}{m\pi} \left[-V_0 \sin \frac{m\pi x}{a} \Big|_{-a}^{-a/2} + V_0 \sin \frac{m\pi x}{a} \Big|_{-a/2}^{+a/2} - V_0 \sin \frac{m\pi x}{a} \Big|_{+a/2}^{+a} \right]$$

$$a_m = \begin{cases} \frac{4V_0}{m\pi} (-1)^n, m = 2n + 1 \text{ (odd)} \\ 0, m = 2, 4, \dots \end{cases}$$

$$f(x) = \frac{4V_0}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left[(2n+1) \frac{\pi x}{a} \right]$$

Solution by "Separation of Variables"

Laplace's equation for an electric potential in Cartesian coordinates

$$\nabla^2 V(\mathbf{x}) = \frac{\partial^2 V(\mathbf{x})}{\partial x^2} + \frac{\partial^2 V(\mathbf{x})}{\partial y^2} + \frac{\partial^2 V(\mathbf{x})}{\partial z^2} = 0$$

Best with rectangular boundary conditions

Solutions are often of the "separable" type,

$V(\mathbf{x}) = X(x)Y(y)Z(z)$ or linear combinations of such solutions:

$$V(\mathbf{x}) = \sum_{n=0}^{\infty} V_n(\mathbf{x});$$
$$V_n(\mathbf{x}) = X_n(x)Y_n(y)Z_n(z)$$

Partial derivatives in Laplace's equation become total derivatives

$$\nabla^2 V(\mathbf{x}) = \frac{d^2 X(x)}{dx^2} YZ + \frac{d^2 Y(y)}{dy^2} XZ + \frac{d^2 Z(z)}{dz^2} XY = 0$$

Divide by V

$$\nabla^2 V(\mathbf{x}) / V(\mathbf{x}) = \frac{1}{X} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y} \frac{d^2 Y(y)}{dy^2} + \frac{1}{Z} \frac{d^2 Z(z)}{dz^2} = 0$$

Coordinate independence in separable solutions

Cartesian coordinate separable solutions

$$V(\mathbf{x}) = X(x)Y(y)Z(z)$$

Translational independence $x \rightarrow x'$

$$V(\mathbf{x}') = X(x')Y(y)Z(z)$$

$$\nabla^2 V / V = \frac{1}{X(x')} \frac{d^2 X(x')}{dx'^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} + \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = 0$$

Values of the last two terms (Y and Z terms) have not changed.

Yet the sum is still ZERO. Each of the 3 terms must be a constant.

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = k_1^2$$

$$\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = k_2^2$$

$$\frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = k_3^2$$

Solution to Laplace's equation must have

$$k_1^2 + k_2^2 + k_3^2 = 0$$

Range of solutions

$$k_1^2 + k_2^2 + k_3^2 = 0$$

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = k_1^2; \quad \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = k_2^2; \quad \frac{1}{Z(z)} \frac{d^2 Z(z)}{dz^2} = k_3^2$$

If $k_1 = 0$ (simple field) function X satisfies

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = 0$$

solution: $X(x) = ax + b$

Constants a & b determined by the boundary conditions

If $k_1^2 > 0$, function $X(x)$ satisfies

$$\frac{d^2 X(x)}{dx^2} = k_1^2 X(x)$$

solution: $X(x) = Ae^{k_1 x} + Be^{-k_1 x}$

etc., for Y & Z

If $k_1^2 < 0$, then $k \rightarrow ik'$ k' real > 0 , and X satisfies

$$\frac{d^2 X(x)}{dx^2} = -k_1'^2 X(x)$$

solution: $X(x) = Ae^{ik_1' x} + Be^{-ik_1' x}$

etc., for Y & Z

Character of solutions

$$k_1^2 > 0$$

$$X(x) = Ae^{k_1 x} + Be^{-k_1 x}$$

Allowable values of A , B , and k_1 determined by symmetries and boundary conditions

If region unbounded $+x$

$$A = 0, \quad X(x) = Be^{-k_1 x}$$

If region unbounded $-x$

$$B = 0, \quad X(x) = Ae^{k_1 x}$$

If region bounded $+ & -$

$$X(x) = A' \cosh k_1 x + B' \sinh k_1 x$$

Even symmetry

Odd symmetry

$$\cosh k_1 x = \left(\frac{e^{k_1 x} + e^{-k_1 x}}{2} \right); \quad \sinh k_1 x = \left(\frac{e^{k_1 x} - e^{-k_1 x}}{2} \right)$$

$$k_1^2 < 0$$

$$X(x) = Ae^{ik_1' x} + Be^{-ik_1' x}$$

Suggests using Fourier series for A , B , k_1

Even symmetry

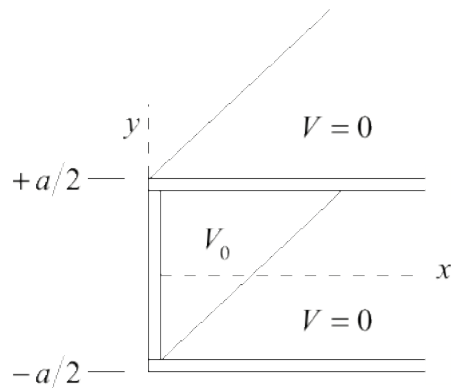
Odd symmetry

$$X(x) = A' \cos k_1' x + B' \sin k_1' x$$

$$\cos k_1 x = \left(\frac{e^{ik_1 x} + e^{-ik_1 x}}{2} \right); \quad \sin k_1 x = \left(\frac{e^{ik_1 x} - e^{-ik_1 x}}{2i} \right)$$

Example: Long narrow channel

Large top and bottom grounded plates, potential V_0 on the end plate
Find potential everywhere between the plates.



$V(x, y) = X(x)Y(y)$ problems without z dependence

all start the same way

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = 0$$

Laplace's equation

Each term must be a constant

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = k_1^2$$

$$\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = k_2^2$$

$$k_1^2 + k_2^2 = 0$$

$$k_1^2 = +k^2, \quad k_2^2 = -k^2$$

Even y symmetry

First determine Y

$$\frac{d^2 Y(y)}{dy^2} = -k^2 Y(y)$$

$$Y(y) = A \cos ky + B \sin ky$$

$$Y(y) = A \cos ky \quad B = 0$$

V = 0 on top and bottom plate

$$Y(a/2) = \cos(ka/2) = 0$$

$$k = (2n+1) \frac{\pi}{a}; \quad n = 0, 1, 2, \dots$$

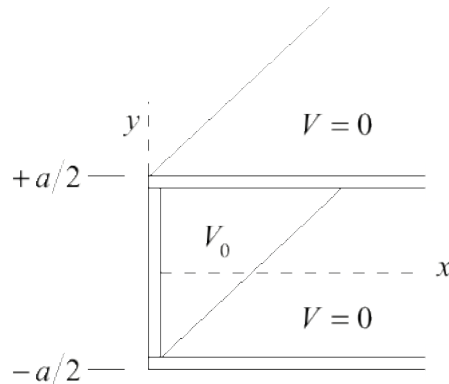
Any one of these values for k, satisfies the boundary condition at top and bottom, But an infinite series is needed to satisfy $V(0, y) = V_0$

Solution with coefficients

$$Y(y) = \sum_{n=0}^{\infty} A_n \cos \left[(2n+1) \frac{\pi y}{a} \right]$$

Example (cont'd)

Determine the X function



$$\frac{d^2 X(x)}{dx^2} = +k^2 X(x)$$

General
solution

$$X(x) = Ae^{kx} + Be^{-kx}$$

Finite at large +x

$$X(x \rightarrow \infty) = 0, \quad A = 0$$

$$X(x) = Be^{-kx}$$

From Y
solution

$$k = (2n+1)\frac{\pi}{a}; \quad n = 0, 1, 2, \dots$$

$$X(x) = Be^{-(2n+1)\pi x/a}$$

V = XY and determine the coefficients

$$C_n = A_n B$$

$$V(x, y) = X(x)Y(y) = \sum_{n=0}^{\infty} C_n \cos \frac{(2n+1)\pi y}{a} e^{-(2n+1)\pi x/a}$$

On left boundary, Potential is V_0

$$V(0, y) = V_0 = \sum_{n=0}^{\infty} C_n \cos \left[\frac{(2n+1)\pi y}{a} \right]$$

Fourier tells us we get C_n this way

$$C_n = \frac{1}{a} \int_{-a}^a V_0 \cos \frac{(2n+1)\pi y}{a} dy = \frac{4V_0 (-1)^n}{(2n+1)\pi}$$

Finally a solution

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \frac{(2n+1)\pi y}{a} e^{-(2n+1)\pi x/a}$$