Problems

26. A rhombus is a parallelogram with opposite sides of equal length. Let us form a rhombus using vectors \( \vec{v}_1 \) and \( \vec{v}_2 \) as two adjacent sides, with \(|\vec{v}_1| = |\vec{v}_2|\). The diagonals of the rhombus are \( \vec{d}_1 = \vec{v}_1 + \vec{v}_2 \) and \( \vec{d}_2 = \vec{v}_1 - \vec{v}_2 \). The angle \( \theta \) between the two diagonals is obtained from \( \cos \theta = \frac{\vec{d}_1 \cdot \vec{d}_2}{|\vec{d}_1||\vec{d}_2|} \). We can obtain this angle from
\[
d_1 \cdot d_2 = (v_1 + v_2) \cdot (v_1 - v_2) = v_1^2 + v_2^2 - v_1 \cdot v_2 - v_1 \cdot v_2 = |v_1|^2 - |v_2|^2 = 0.
\]
So \( \cos \theta = 0 \) and the diagonals are orthogonal.

27. \( \epsilon_{ijk} \epsilon_{ijl} = (\epsilon_{123})^2 + (\epsilon_{132})^2 + (\epsilon_{213})^2 + (\epsilon_{231})^2 + (\epsilon_{312})^2 + (\epsilon_{321})^2 = 1 + 1 + 1 + 1 + 1 + 1 = 6. \)
(All other \( \epsilon_{ijk} \)'s are 0.) Or
\[
\epsilon_{ijk} \epsilon_{ijl} = \delta_{d} \delta_{jmc} (\epsilon_{ijk} \epsilon_{imn}) = \delta_{d} \delta_{jmc} (\delta_{d} \delta_{jm} - \delta_{im} \delta_{jl}) = 3 \cdot 3 - 3 = 6.
\]

28. (a) \([\langle A \cdot \nabla \rangle B + \langle B \cdot \nabla \rangle A + A \times (\nabla \times B) + B \times (\nabla \times A) \]_i = A_j \partial_j B_i + B_j \partial_j A_i + \epsilon_{ijk} A_j (\epsilon_{kml} \partial_l B_m) + \epsilon_{ijk} B_j (\epsilon_{kml} \partial_l A_m) = A_j \partial_j B_i + B_j \partial_j A_i + \epsilon_{ijk} \epsilon_{klm} (A_j \partial_l B_m + B_j \partial_l A_m) = A_j \partial_j B_i + B_j \partial_j A_i + (\delta_{d} \delta_{jm} - \delta_{im} \delta_{jl}) (A_j \partial_l B_m + B_j \partial_l A_m) = A_j \partial_j B_i + B_j \partial_j A_i + (A_j \partial_l B_j + B_j \partial_l A_j - A_j \partial_l B_i - B_j \partial_l A_i) = A_j \partial_j B_j + B_j \partial_j A_j = \partial_i (A_j B_j) \) (chain rule for derivatives)
\[
= [\nabla (A \cdot B)]_i.
\]
(b) \( V \cdot (\nabla \times U) - U \cdot (\nabla \times V) \)
\[
= V_i \epsilon_{ijk} \partial_j U_k - U_i \epsilon_{ijk} \partial_j V_k = \epsilon_{kij} V_k \partial_j U_j - \epsilon_{ijk} U_j \partial_j V_k \) (just re-labeling indices)
\[
= \epsilon_{ijk} V_k \partial_j U_j + \epsilon_{ijk} U_j \partial_j V_k \) (antisymmetry of \( \epsilon_{ijk} \))
\[
= \delta_i (\epsilon_{ijk} U_j V_k) \) (chain rule for derivatives)
\[
= \partial_i (\epsilon_{ijk} U_j V_k)
\]
\[
= \nabla \cdot (U \times V).
\]

29. We can write the charge distribution as \( \rho(\vec{r}) = -q \delta^{(3)}(\vec{r}) + \frac{q}{2 \pi R} \delta(z) \delta(r - R). \)
The \( \delta^{(3)}(\vec{r}) \) is defined so \( \int \delta^{(3)} d^3 r(\vec{r}) = 1 \) if the integral includes the origin, and the second term puts the remaining charge in the \( xy \) plane \( (\delta(z)) \) at radius \( R \) \( (\delta(r - R)) \) with appropriate normalization. I am using cylindrical coordinates \((r, \theta, z)\). We calculate the first 3 electric multipole moments.

(a) Monopole: \( Q = \int d^3 r \rho(\vec{r}) = -q + \frac{q}{2 \pi R} \int dz \int d \theta d \delta(z) \delta(r - R) = -q + \frac{q}{2 \pi R} (2 \pi R) = 0. \)
This verifies that we have the correct normalization for the second term of \( \rho(\vec{r}) \).
(b) Dipole: By symmetry, the only possible direction that this could point in is the $z$ direction. But $Q_3 = \int d^3r \rho(\vec{r}) z = 0$, since both terms in $\rho(\vec{r})$ contain $\delta(z)$. (All of the charge lies in the plane $z = 0$.) Thus, we expect $(Q_1, Q_2, Q_3) = (0, 0, 0)$. Let us check for $Q_1$:

$$Q_1 = \int d^3r \rho(\vec{r}) x = \int d^3r \rho(\vec{r}) r \cos \theta = -\frac{qR}{2\pi R} \int dz r dr d\theta \delta(z) \delta(r - R) r \cos \theta = \frac{qR}{2\pi} \int_0^{2\pi} \theta \cos \theta = 0.$$ 

So we have $Q_1 = Q_2 = Q_3 = 0$.

(c) Quadrupole: Use $Q_{ij} = \frac{1}{2} \int d^3r \rho(\vec{r})(3x_i x_j - \vec{r}^2 \delta_{ij})$. By symmetry of $\rho(\vec{r})$ and the fact that $Q_{ij}$ is symmetric and traceless, we expect $Q_{12} = Q_{21}$, $Q_{13} = Q_{31} = Q_{23} = Q_{32}$, $Q_{11} = Q_{22}$, and $Q_{33} = -Q_{11} - Q_{22}$. Thus, we only need to calculate three components:

$$Q_{12} = \frac{1}{2} \frac{qR^2}{2\pi R} \int dz r dr d\theta \delta(z) \delta(r - R) 3(r \cos \theta)(r \sin \theta) = \frac{3}{2} \frac{qR^2}{2\pi} \int_0^{2\pi} \theta \sin \theta \cos \theta = 0.$$ 

$$Q_{13} = \frac{1}{2} \frac{qR^2}{2\pi R} \int dz r dr d\theta \delta(z) \delta(r - R) 3(r \cos \theta)(z) = 0.$$ 

$$Q_{11} = \frac{1}{2} \frac{qR^2}{2\pi R} \int dz r dr d\theta \delta(z) \delta(r - R) [3(r \cos \theta)^2 - (r^2 + z^2)] = \frac{1}{2} \frac{qR^2}{2\pi} \int_0^{2\pi} \theta [3 \cos^2 \theta - 1] = \frac{1}{2} \frac{qR^2}{2\pi} (\pi) = \frac{3}{4}.$$ 

Although we know that $Q_{33} = -2Q_{11} = -2Q_{22}$ by symmetry and tracelessness, let us calculate it anyway as a check:

$$Q_{33} = \frac{1}{2} \frac{qR^2}{2\pi R} \int dz r dr d\theta \delta(z) \delta(r - R) [3z^2 - (r^2 + z^2)] = -\frac{1}{2} \frac{qR^2}{2\pi} \int_0^{2\pi} \theta = -\frac{qR^2}{2}.$$ 

Thus, we get $Q_{11} = Q_{22} = \frac{qR^2}{4}$, $Q_{33} = -\frac{qR^2}{2}$, and all other $Q_{ij} = 0$.

30. We need the most general four-index, symmetric and traceless tensor that we can build out of $x_i$, $\delta_{ij}$, and $\epsilon_{ijk}$. Of course, since it is symmetric, we can’t use $\epsilon_{ijk}$. So there are only three possible types of terms: $x_i x_j x_k x_l$, $\delta_{ij} x_k x_l$, and $\delta_{ij} \delta_{kl}$. Of these, the first term is already symmetric under interchange of any indices. To make the other terms symmetric, we must add all non-identical permutations of indices. In this way our tensor must be of the form:

$$X_{ijkl} = x_i x_j x_k x_l + b [\delta_{ij} x_k x_l + \delta_{ik} x_j x_l + \delta_{il} x_k x_j + \delta_{jk} x_i x_l + \delta_{jl} x_i x_k + \delta_{kl} x_i x_j] + c [\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}],$$

where $b$ and $c$ are coefficients to be determined, and an overall normalization factor $a$ has already been pulled out. Now we just need to make this traceless, by requiring $\delta_{ij} X_{ijkl} = 0$. (Tracing on any two indices will give the same answer, since it has already been made symmetric.) This gives

$$0 = \vec{r}^2 x_i x_l + b [3x_i x_l + x_k x_l + x_x x_l + x_k x_l + \delta_{kl} \vec{r}^2] + c [3\delta_{kl} + \delta_{kl} + \delta_{kl}] = x_k x_l (\vec{r}^2 + 7b) + \delta_{kl} (\vec{r}^2 b + 5c).$$

Since this must be zero for any $\vec{r}$, we must have the coefficients of $x_k x_l$ and $\delta_{kl}$ both equal to zero. We get $b = -\vec{r}^2 / 7$ and $c = +(\vec{r}^2)/35$, so

$$X_{ijkl} = x_i x_j x_k x_l - \frac{1}{35} \vec{r}^2 [\delta_{ij} x_k x_l + \delta_{ik} x_j x_l + \delta_{il} x_k x_j + \delta_{jk} x_i x_l + \delta_{jl} x_i x_k + \delta_{kl} x_i x_j] + \frac{1}{35} (\vec{r}^2)^2 [\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}].$$
31. In cylindrical coordinates, \((q_1,q_2,q_3) = (\rho, \theta, z)\), a position in three space is 
\( \vec{r} = (x, y, z) = (\rho \cos \theta, \rho \sin \theta, z) \).

(a) The curvilinear unit vectors are obtained from
\[
\begin{align*}
\frac{\partial \vec{e}}{\partial \rho} &= (\cos \theta, \sin \theta, 0) = \hat{e}_\rho \\
\frac{\partial \vec{e}}{\partial \theta} &= (-\rho \sin \theta, \rho \cos \theta, 0) = \rho \hat{e}_\theta \\
\frac{\partial \vec{e}}{\partial z} &= (0, 0, 1) = \hat{e}_z, \\
\end{align*}
\]
so we have \( h_\rho = 1, h_\theta = \rho, h_z = 1 \) and
\[
\begin{align*}
\hat{e}_\rho &= (\cos \theta, \sin \theta, 0) \\
\hat{e}_\theta &= (-\sin \theta, \cos \theta, 0) \\
\hat{e}_z &= (0, 0, 1) \\
\end{align*}
\]

(b) In cylindrical coordinates:
\[
\vec{\nabla} \cdot \vec{V} = \frac{1}{1 + \rho^2} \left[ \frac{\partial}{\partial \rho} (V_\rho \rho + 1) + \frac{\partial}{\partial \theta} (V_\theta + 1) + \frac{\partial}{\partial z} (V_z \rho) \right]
= \frac{1}{\rho} \frac{\partial (\rho V_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}.
\]

(c) Now for \( \vec{\nabla} \times \vec{V} \):
\[
\begin{align*}
[\vec{\nabla} \times \vec{V}]_\rho &= \frac{1}{1 + \rho^2} \left[ \frac{\partial}{\partial \theta} (V_z - \rho V_\theta) \right] = \frac{1}{\rho} \left[ \frac{\partial V_z}{\partial \theta} - \frac{\partial (\rho V_\theta)}{\partial \theta} \right]. \\
[\vec{\nabla} \times \vec{V}]_\theta &= \frac{1}{1 + \rho^2} \left[ \frac{\partial}{\partial z} (V_\rho - \rho V_\theta) \right] = \frac{1}{\rho} \left[ \frac{\partial V_\rho}{\partial z} - \frac{\partial (\rho V_\theta)}{\partial z} \right]. \\
[\vec{\nabla} \times \vec{V}]_z &= \frac{1}{1 + \rho^2} \left[ \frac{\partial}{\partial \rho} (\rho V_\theta - \rho V_\theta) \right] = \frac{1}{\rho} \left[ \frac{\partial (\rho V_\theta)}{\partial \rho} - \frac{\partial V_\theta}{\partial \theta} \right]. \\
\end{align*}
\]

(d) Fluid flows in a pipe of radius \( R \) with a velocity field given by
\[
\vec{V} = V_0 (1 - \rho^2/R^2) \hat{e}_z. \
\]
So \( V_\rho = V_\theta = 0 \) and \( V_z = V_0 (1 - \rho^2/R^2) \).
The divergence of this field is
\[
\vec{\nabla} \cdot \vec{V} = \frac{\partial V_z}{\partial z} = 0.
\]
The curl of this field is
\[
\vec{\nabla} \times \vec{V} = \frac{1}{\rho} \frac{\partial V_\theta}{\partial \rho} \hat{e}_\rho - \frac{\partial V_z}{\partial \theta} \hat{e}_\theta = \frac{2V_0 \rho}{R^2} \hat{e}_\theta.
\]

32. The position vector \( \vec{r}(\xi, \eta, \phi) = (x, y, z) = (u(\xi, \eta) \cos \phi, u(\xi, \eta) \sin \phi, v(\xi, \eta)) \).

(a) \[
\begin{align*}
\frac{\partial \vec{e}}{\partial \xi} &= \frac{\partial u}{\partial \xi} (\cos \phi, \sin \phi, 0) + \frac{\partial v}{\partial \xi} (0, 0, 1) \\
\frac{\partial \vec{e}}{\partial \eta} &= \frac{\partial u}{\partial \eta} (\cos \phi, \sin \phi, 0) + \frac{\partial v}{\partial \eta} (0, 0, 1) \\
\frac{\partial \vec{e}}{\partial \phi} &= u(- \sin \phi, \cos \phi, 0). \\
\end{align*}
\]

(b) The vector \( \frac{\partial \vec{e}}{\partial \phi} \) is easily seen to be orthogonal to the other two vectors.
Thus we only need to check \( \frac{\partial \vec{e}}{\partial \xi} \cdot \frac{\partial \vec{e}}{\partial \eta} = \frac{\partial u}{\partial \xi} \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \xi} \frac{\partial v}{\partial \eta} \).
If \( u \) and \( v \) satisfy the Cauchy-Riemann equations,
\[
\frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial \eta}, \quad \frac{\partial v}{\partial \xi} = -\frac{\partial u}{\partial \eta},
\]
then \( \frac{\partial \vec{e}}{\partial \xi} \cdot \frac{\partial \vec{e}}{\partial \eta} = \left( \frac{\partial v}{\partial \eta} \right) \frac{\partial u}{\partial \eta} + \left( -\frac{\partial u}{\partial \eta} \right) \frac{\partial v}{\partial \eta} = 0. \)
The vectors are orthogonal.

(c) Now let \( u(\xi, \eta) = \xi \eta \) and \( v(\xi, \eta) = (\xi^2 - \eta^2)/2 \) (parabolic coordinates).
Then
34. If the set of vectors \( \{x_i\} \) gives a basis \( V \) spanning the set \( \{c_i x_i\} \), where \( c_i \) are nonzero scalars, then we want to prove that the set \( \{c_ix_i\} \) is also a basis for \( V \). To do this we must show that \( \sum_{i=1}^{n} \alpha_i x_i = 0 \) implies \( \alpha_i = 0 \) for all \( i \). So, let’s assume \( \sum_{i=1}^{n} \alpha_i x_i = 0 \). This means \( \sum_{i=1}^{n} \alpha_i x_i = (\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \sum_{i=1}^{n-1} \alpha_i, \sum_{i=1}^{n} \alpha_i) = (0, 0, 0, \ldots, 0). \)

Looking at the first position of the \( n \)-tuple gives \( \alpha_1 = 0 \). The second position gives \( \alpha_1 + \alpha_2 = 0 \), which leads to \( \alpha_2 = 0 \). Continuing by induction, we see that each of the \( \alpha_i = 0 \). Thus, the \( n \)-tuples are linearly independent.

33. We want to prove that:

\[
\begin{align*}
x_1 &= (1, 1, 1, \ldots, 1, 1) \\
x_2 &= (0, 1, 1, \ldots, 1, 1) \\
x_3 &= (0, 0, 1, \ldots, 1, 1) \\
&\vdots \\
x_n &= (0, 0, 0, \ldots, 0, 1)
\end{align*}
\]

are linearly independent. To do this we must show that \( \sum_{i=1}^{n} \alpha_i x_i = 0 \) implies all \( \alpha_i = 0 \). So, let’s assume \( \sum_{i=1}^{n} \alpha_i x_i = 0 \). This means \( \sum_{i=1}^{n} \alpha_i x_i = (\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \sum_{i=1}^{n-1} \alpha_i, \sum_{i=1}^{n} \alpha_i) = (0, 0, 0, \ldots, 0, 0). \)

Looking at the first position of the \( n \)-tuple gives \( \alpha_1 = 0 \). The second position gives \( \alpha_1 + \alpha_2 = 0 \), which leads to \( \alpha_2 = 0 \). Continuing by induction, we see that each of the \( \alpha_i = 0 \). Thus, the \( n \)-tuples are linearly independent.

1) Linear independence:

Assume \( \sum_{i=1}^{n} \alpha_i (c_i x_i) = 0 \). This also can be written \( \sum_{i=1}^{n} (\alpha_i c_i) x_i = 0 \). Since the set \( \{x_i\} \) are linearly independent, this must imply each of the \( \alpha_i c_i = 0 \). Since we are given that each of the \( c_i \neq 0 \), this also implies that each of the \( \alpha_i = 0 \). Therefore, \( \sum_{i=1}^{n} \alpha_i (c_i x_i) = 0 \) only if all \( \alpha_i = 0 \). Therefore, the set \( \{c_ix_i\} \) must be linearly independent.

2) It spans \( V \):

Assume \( x \in V \). Then we can write \( x = \sum_{i=1}^{n} \beta_i x_i \) for some scalars \( \beta_i \). Since the \( c_i \neq 0 \), we can multiply and divide each term by \( c_i \) to get
\[ x = \sum_{i=1}^{n} \beta_i'(c_i x_i), \text{ where } \beta_i' = \beta_i/c_i. \] Thus, every vector can be written as a linear combination of the \{c_i x_i\}. The set spans \( V \).

A geometrical interpretation is that the length of the basis vectors can be scaled, and it will still be a good basis.