Physics for Scientists & Engineers 2

Spring Semester 2005

Lecture 7

Gauss' Law for Various Charge Distributions

- We have applied Gauss' Law to a point charge and showed that we get Coulomb's Law
- Now let's look at more complicated distributions of charge and calculate the resulting electric field
- We will use the quantity charge density to describe the distribution of charge
- This charge density will be different depending on the geometry

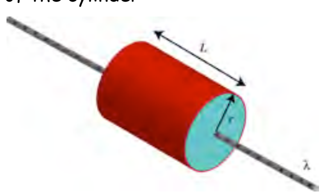
Symbol	Name	Unit
λ	Charge per length	C/m
σ	Charge per area	C/m^2
ρ	Charge per volume	C/m^3

Cylindrical Symmetry

- Let's calculate the electric field from a conducting wire with charge per unit length λ using Gauss' Law

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = q$$

- We start by assuming a Gaussian surface in the form of a right cylinder with radius r and length L placed around the wire such that the wire is along the axis of the cylinder



Cylindrical Symmetry (2)

- From symmetry we can see that the electric field will extend radially from the wire
- How?
  - If we rotate the wire along its axis, the electric field must look the same
    - Cylindrical symmetry
  - If we imagine a very long wire, the electric field cannot be different anywhere along the length of the wire
    - Translational symmetry
- Thus our assumption of a right cylinder as a Gaussian surface is perfectly suited for the calculation of the electric field using Gauss' Law

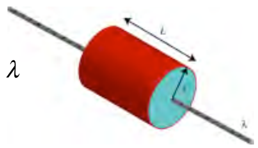
Cylindrical Symmetry

- The contribution to our integral from the caps of the cylinder is zero because the electric field is always parallel to the caps
- The electric field is always perpendicular to the wall of the cylinder so we can write

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 EA = \varepsilon_0 E(2\pi rL) = q = \lambda L$$

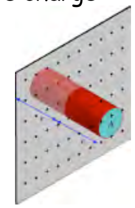
- Rewriting we get the electric field at a distance *r* from a conducting wire with charge density λ

$$E = \frac{\lambda}{2\pi\varepsilon_0 r} = \frac{2k\lambda}{r}$$



Planar Symmetry

- Assume that we have a thin, infinite non-conducting sheet of positive charge



- The charge density in this case is the charge per unit area, σ
- From symmetry, we can see that the electric field will be perpendicular to the surface of the sheet

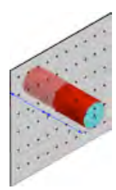
Planar Symmetry (2)

- To calculate the electric field using Gauss' Law, we assume a Gaussian surface in the form of a right cylinder with cross sectional area *A* and height 2*r*, chosen to cut through the plane perpendicularly
- Because the electric field is perpendicular to the plane everywhere, the electric field will be parallel to the walls of the cylinder and perpendicular to the caps of the cylinder
- Using Gauss' Law we get

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 (EA + EA) = q = \sigma A$$

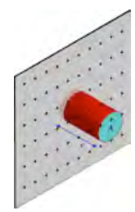
- Which gives us the electric field from an infinite non-conducting sheet with charge density σ

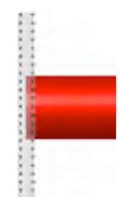
$$E = \frac{\sigma}{2\varepsilon_0}$$



Planar Symmetry (3)

- Assume that we have a thin, infinite conductor with positive charge





- The charge density in this case is also the charge per unit area, σ
- From symmetry, we can see that the electric field will be perpendicular to the surface of the sheet

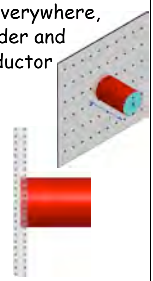
Planar Symmetry (4)

- To calculate the electric field using Gauss' Law, we assume a Gaussian surface in the form of a right cylinder with cross sectional area A and height r, chosen to cut through one side of the plane perpendicularly
- The field inside the conductor is zero so the cap inside the conductor does not contribute to the integral
- Because the electric field is perpendicular to the plane everywhere, the electric field will be parallel to the walls of the cylinder and perpendicular to the cap of the cylinder outside the conductor
- Using Gauss' Law we get

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 (EA) = q = \sigma A$$

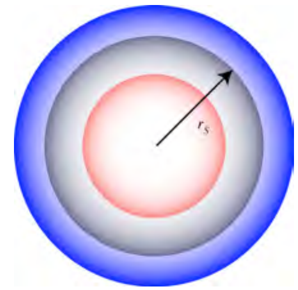
- Which gives us the electric field from an infinite conducting sheet with charge density σ

$$E = \frac{\sigma}{\varepsilon_0}$$



Spherical Symmetry

- Now let's calculate the electric field from charge distributed as a spherical shell
- Assume that we have a spherical shell of charge q with radius r_s (gray)
- We will assume two different spherical Gaussian surfaces
  - $r > r_s$ (blue)
  - $r < r_s$ (red)



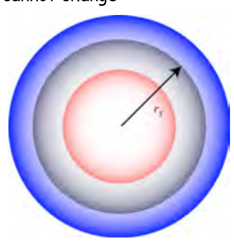
Spherical Symmetry (2)

- Let's start with the Gaussian surface outside the sphere of charge, $r > r_s$ (blue)
- We know from symmetry arguments that the electric field will be radial outside the charged sphere
  - If we rotate the sphere, the electric field cannot change
    - Spherical symmetry
- Thus we can apply Gauss' Law and get

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 EA = \varepsilon_0 E\left(4\pi r^2\right) = q$$

- Which we can rewrite as

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$



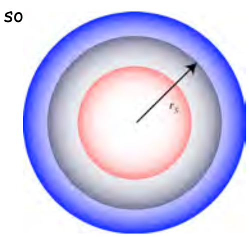
Spherical Symmetry (3)

- Let's let's take the Gaussian surface inside the sphere of charge, $r < r_s$ (red)
- We know that the enclosed charge is zero so

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 E\left(4\pi r^2\right) = 0$$

- And we get the result that the electric field is zero everywhere inside the the spherical shell of charge

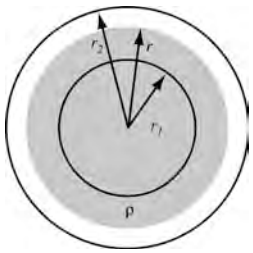
$$E = 0$$



- Thus we obtain two results
  - The electric field outside a spherical shell of charge is the same as that of a point charge
  - The electric field inside a spherical shell of charge is zero

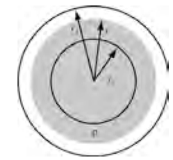
Spherical Symmetry (4)

- Now let's calculate the electric field from charge distributed uniformly throughout a sphere
- Assume that we have a solid sphere of charge q with radius r with constant charge density per unit volume ρ
- We will assume two different spherical Gaussian surfaces
  - $r_2 > r$
  - $r_1 < r$



Spherical Symmetry (5)

- Let's start with a Gaussian surface with $r_1 < r$
- From spherical symmetry we know that the electric field will be radial and perpendicular to the Gaussian surface
- Gauss' Law gives us



$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 E\left(4\pi r_1^2\right) = q = \rho\left(\frac{4}{3}\pi r_1^3\right)$$

$4\pi r_1^2$ is the area of the Gaussian surface

$\frac{4}{3}\pi r_1^3$ is the volume enclosed by the Gaussian surface

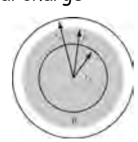
- We can rewrite this equation as

$$E = \frac{\rho r_1}{3\varepsilon_0}$$

Spherical Symmetry (6)

- We have a result as a function of the radius of our assumed Gaussian surface, but we really need the result in terms of the total charge

$$q_t = \rho \frac{4}{3}\pi r^3$$



- The charge enclosed by the assumed Gaussian surface is

$$q = \frac{\text{Volume inside } r_1}{\text{Volume of charge distribution}} q_t = \frac{\frac{4}{3}\pi r_1^3}{\frac{4}{3}\pi r^3} q_t = q_t \frac{r_1^3}{r^3}$$

- Which we can substitute back into the expression we obtained from Gauss's Law to get

$$E(r_1) = \frac{q_t r_1}{4\pi\varepsilon_0 r^3} = \frac{k q_t r_1}{r^3}$$

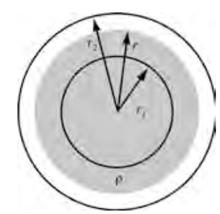
Spherical Symmetry (5)

- Now let's look at a Gaussian surface with $r_2 > r$
- From spherical symmetry we know that the electric field will be radial and perpendicular to the Gaussian surface
- Gauss' Law gives us

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = \varepsilon_0 E\left(4\pi r_2^2\right) = q_t$$

$4\pi r_2^2$ is the area of the Gaussian surface

q_t is the total charge of the sphere



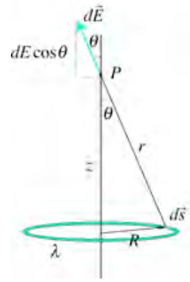
- We can rewrite this equation as

$$E(r_2) = k\frac{q_t}{r_2^2} \text{ or just } E = k\frac{q_t}{r^2}$$

- Which is the same as the field from a point charge

Electric Field from a Ring of Charge

- Let's now calculate the electric field resulting from a ring of charge
  - The ring has a radius R
  - The ring lies in the x-y plane such that z-axis is perpendicular to the plane of the ring and the origin is at the center of the ring
  - The ring has a linear charge density λ and total charge q
- What is the electric field along the z-axis?



Electric Field from a Ring of Charge (2)

- The differential charge due to a differential arc length is

$$dq = \lambda ds$$

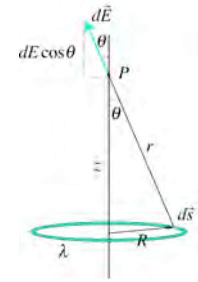
- The differential electric field at a point P along the z-axis is given by

$$dE = k\frac{dq}{r^2} = k\frac{\lambda ds}{R^2 + z^2}$$

- The component of the electric field along z-axis is given by

$$dE\cos\theta$$

$$\cos\theta = \frac{z}{r} = \frac{z}{\sqrt{R^2 + z^2}}$$



Electric Field from a Ring of Charge (3)

- Now we need to integrate the contribution to the electric field along the z-axis around the ring

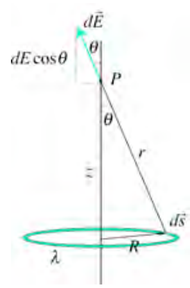
$$E = \int dE\cos\theta$$

- Substituting in our expression for $\cos\theta$ we get

$$E = \int dE\frac{z}{\sqrt{R^2 + z^2}} = \frac{z}{\sqrt{R^2 + z^2}}\int dE$$

- Adding our expression for dE we get

$$E = \frac{z}{\sqrt{R^2 + z^2}}\int k\frac{\lambda ds}{R^2 + z^2} = \frac{z\lambda}{\left(R^2 + z^2\right)^{3/2}}\int_0^{2\pi R} ds$$



Electric Field from a Ring of Charge (4)

- Our expression then becomes

$$E = \frac{kz\lambda(2\pi R)}{\left(R^2 + z^2\right)^{3/2}}$$

- Remembering that

$$q = \lambda(2\pi R)$$

- We get the electric field along the z-axis

$$E = k\frac{zq}{\left(R^2 + z^2\right)^{3/2}}$$

- If we are far away from the ring ($z \gg R$)

$$E = k\frac{q}{z^2}$$