Physics for Scientists & Engineers 2

Spring Semester 2005

Lecture 28

Electromagnetic Oscillations

- We have been working with circuits that have
  - a constant current
  - a current that increases to a constant current
  - a current that decreases to zero current
- Now we will introduce circuits that have resistors, capacitors, and inductors as well as time-varying sources of emf
- These types of circuits display different phenomena than the circuits we have been studying, e.g.
  - Resonance
  - Transformers

Energy in Inductors

- Consider a circuit with a resistor *R* and a capacitor *C*
  - The current increases or decreases exponentially with a time constant $\tau_C = RC$
- Consider a circuit with an inductor *L* and a capacitor *C*
  - The current increases or decreases exponentially with a time constant $\tau_L = L/R$
- Now consider a circuit with an inductor *L* and a capacitor *C*
  - We will see that the current and voltage will vary sinusoidally with time
- We call these variations in voltage and current electromagnetic oscillations

Energy in Inductors (2)

- Remember that the energy stored in the electric field of a capacitor with charge *q* is given by

$$U_E = \frac{1}{2}\frac{q^2}{C}$$

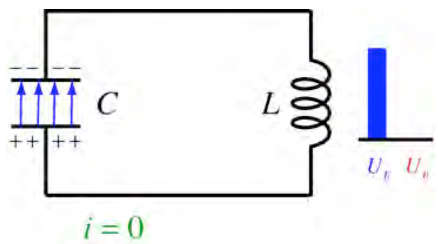
- Remember that the energy stored in the magnetic field of an inductor carrying current *i* is given by

$$U_B = \frac{1}{2}Li^2$$

- To illustrate electromagnetic oscillations, consider a simple circuit containing a capacitor *C* and an inductor *L*

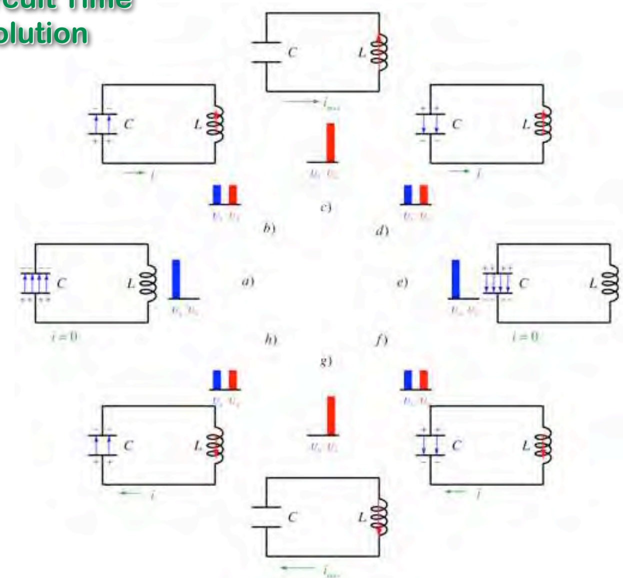
LC Circuit

- The capacitor is initially fully charged and then connected to the circuit
- The energy in the circuit resides solely in the electric field of the capacitor
- The current is zero



- Now let's follow the evolution with time of the current, charge, magnetic energy, and electric energy in the circuit

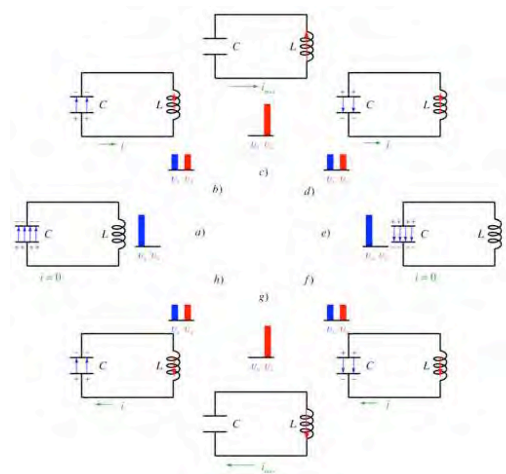
LC Circuit Time Evolution

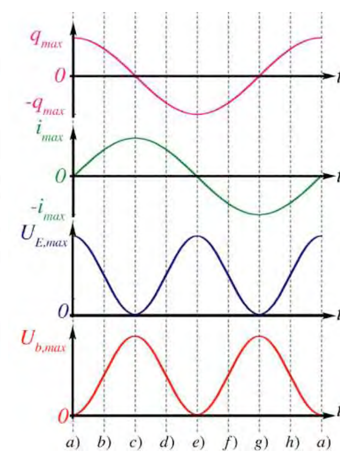


LC Circuits (2)

- The charge on the capacitor varies with time
  - Max positive to zero to max negative to zero back to max positive
- The current in the inductor varies with time
  - Zero to max positive to zero to max negative back to zero
- The energy in the inductor varies with the square of the current and the energy in the capacitor varies with the square of the charge
  - The energies vary between zero and a maximum value

LC Circuits (3)





LC Oscillations

- Now we derive a quantitative description of the phenomena we just described qualitatively
- We assume a single loop circuit containing a capacitor *C* and an inductor *L* and that there is no resistance in the circuit
- We can write the energy in the circuit *U* as the sum of the electric energy in the capacitor and the magnetic energy in the inductor

$$U = U_E + U_B$$

- We can re-write the electric and magnetic energies in terms of the charge *q* and current *i*

$$U = U_E + U_B = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}Li^2$$

LC Oscillations (2)

- Because we have assumed that there is no resistance and the electric field and magnetic field are conservative, the energy in the circuit will be constant
- Thus the derivative of the energy in the circuit with respect to time will be zero
- We can then write

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}\frac{q^2}{C} + \frac{1}{2}Li^2\right) = \frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} = 0$$

- Remembering that $i = dq/dt$ we can write

$$\frac{di}{dt} = \frac{d}{dt}\left(\frac{dq}{dt}\right) = \frac{d^2q}{dt^2}$$

LC Oscillations (3)

- We can then write

$$\frac{q}{C}\frac{dq}{dt} + L\frac{dq}{dt}\frac{d^2q}{dt^2} = \frac{q}{C} + L\frac{d^2q}{dt^2} = 0$$

- Which we can rewrite as

$$\frac{d^2q}{dt^2} + \frac{q}{LC} = 0$$

- This differential equation has the same form as that of simple harmonic motion describing the position *x* of an object with mass *m* connected to a spring with spring constant *k*

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

LC Oscillations (4)

- The solution to the differential equation describing simple harmonic motion was

$$x = x_{\max}\cos(\omega_0 t + \phi)$$

- where ϕ is the phase and ω_0 is the angular frequency

$$\omega_0 = \sqrt{\frac{k}{m}}$$

- By analogy we can get the charge as a function of time

$$q = q_{\max}\cos(\omega_0 t + \phi)$$

- where ϕ is the phase and ω_0 is the angular frequency

$$\omega_0 = \sqrt{\frac{1}{LC}} = \frac{1}{\sqrt{LC}}$$

LC Oscillations (5)

- The current is then given by

$$i = \frac{dq}{dt} = \frac{d}{dt}\left(q_{\max}\cos\left(\omega_0 t + \phi\right)\right) = -\omega_0 q_{\max}\sin\left(\omega_0 t + \phi\right)$$

- Realizing that the magnitude of the maximum current in the circuit is given by $\omega_0 q_{max}$ we get

$$i = -i_{\max}\sin(\omega_0 t + \phi)$$

- Having expression for the charge as a function of time we can write and expression for the electric energy

$$U_E = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}\frac{\left(q_{\max}\cos\left(\omega_0 t + \phi\right)\right)^2}{C} = \frac{q_{\max}^2}{2C}\cos^2\left(\omega_0 t + \phi\right)$$

LC Oscillations (6)

- Having expression for the current as a function of time we can write and expression for the magnetic energy

$$U_B = \frac{1}{2}Li^2 = \frac{L}{2}\left(-i_{\max}\sin(\omega_0 t + \phi)\right)^2 = \frac{L}{2}i_{\max}^2\sin^2(\omega_0 t + \phi)$$

- Remembering that

$$i_{\max} = \omega_0 q_{\max} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

- We can write

$$\frac{L}{2}i_{\max}^2 = \frac{L}{2}\omega_0^2 q_{\max}^2 = \frac{q_{\max}^2}{2C}$$

LC Oscillations (7)

- Now we can write an expression for the magnetic energy as a function of time

$$U_B = \frac{q_{\max}^2}{2C}\sin^2(\omega_0 t + \phi)$$

- We can write an expression for the energy in the4 circuit by adding the electric energy and the magnetic energy

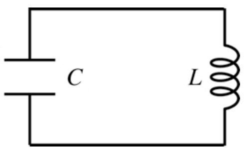
$$U = U_E + U_B = \frac{q_{\max}^2}{2C}\cos^2\left(\omega_0 t + \phi\right) + \frac{q_{\max}^2}{2C}\sin^2(\omega_0 t + \phi)$$

$$U = \frac{q_{\max}^2}{2C}\left(\sin^2(\omega_0 t + \phi) + \cos^2\left(\omega_0 t + \phi\right)\right) = \frac{q_{\max}^2}{2C}$$

- Thus the energy in the circuit remains constant with time and is proportional to the square of the original charge put on the capacitor

LC Circuit Example

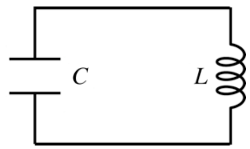
- Consider a circuit containing a capacitor C = 1.50 μF and an inductor L = 3.50 mH. The capacitor is fully charged using a battery with V_{emf} = 12.0 V and then connected to the circuit.



- What is the oscillation frequency of the circuit?
- What is the energy stored in the circuit?
- What is the charge on the capacitor after t = 2.50 s?

LC Circuit Example (2)

- The oscillation frequency of the circuit is given by



$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\left(3.50 \cdot 10^{-3}\ \mathrm{H}\right)\left(1.50 \cdot 10^{-6}\ \mathrm{F}\right)}}$$

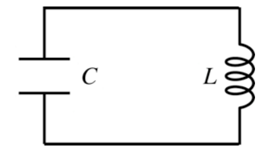
$$\omega_0 = 1.38 \cdot 10^{5}\ \mathrm{Hz}$$

- The energy stored in the circuit is given by

$$U = \frac{q_{max}^2}{2C}$$

LC Circuit Example (3)

- The maximum charge on the capacitor is given by



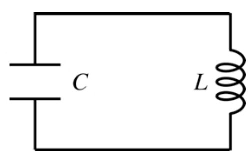
$$q_{max} = CV_{emf} = \left(1.50 \cdot 10^{-6}\ \mathrm{F}\right)\left(12.0\ \mathrm{V}\right)$$

$$q_{max} = 1.80 \cdot 10^{-5}\ \mathrm{C}$$

$$U = \frac{q_{max}^2}{2C} = \frac{\left(1.80 \cdot 10^{-5}\ \mathrm{C}\right)^2}{2 \cdot 1.50 \cdot 10^{-6}\ \mathrm{F}} = 1.08 \cdot 10^{-4}\ \mathrm{J}$$

LC Circuit Example (4)

- The charge on the capacitor as a function of time is given by



$$q = q_{max} \cos\left(\omega_0 t + \phi\right)$$

$$\text{at } t = 0,\ q = q_{max} \quad \Rightarrow \quad \phi = 0$$

$$q = q_{max} \cos\left(\omega_0 t\right)$$

$$q_{max} = 1.80 \cdot 10^{-5}\ \mathrm{C}$$

$$\omega_0 = 1.38 \cdot 10^{5}\ \mathrm{Hz}$$

$$\text{at } t = 2.50\ \mathrm{s}, \text{ we have}$$

$$q = \left(1.80 \cdot 10^{-5}\ \mathrm{C}\right)\cos\left(\left[1.38 \cdot 10^{5}\ \mathrm{Hz}\right]\left[2.50\ \mathrm{s}\right]\right)$$

$$q = \text{-}1.21 \cdot 10^{-5}\ \mathrm{C}$$