Physics for Scientists & Engineers 2

Spring Semester 2005

Lecture 32

Review

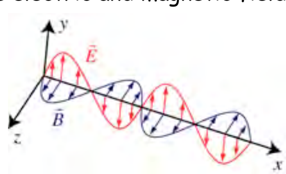
- Maxwell's Equations

Name	Equation	Description
Gauss' Law for Electric Fields	$\oint \vec{E} \bullet d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$	Relates the net electric flux to the net enclosed electric charge
Gauss' Law for Magnetic Fields	$\oint \vec{B} \bullet d\vec{A} = 0$	States that the net magnetic flux is zero (no magnetic charge)
Faraday's Law	$\oint \vec{E} \bullet d\vec{s} = -\frac{d\Phi_B}{dt}$	Relates the induced electric field to the changing magnetic flux
Ampere-Maxwell Law	$\oint \vec{B} \bullet d\vec{s} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} + \mu_0 i_{enc}$	Relates the induced magnetic field to the changing electric flux and to the current

Review (2)

- We make the *Ansatz* that the magnitude of the electric and magnetic fields in electromagnetic waves are given by the form

$$E(\vec{r},t) = E_{\max} \sin(kx - \omega t)$$
$$B(\vec{r},t) = B_{\max} \sin(kx - \omega t)$$



- where $k = 2\pi/\lambda$ is the angular wave number and $\omega = 2\pi f$ is the angular frequency of a wave with wavelength λ and frequency f
- We assume that the electric field is in the y direction and the magnetic field is in the z direction

$$\vec{E}(\vec{r},t) = E(\vec{r},t)\vec{e}_y$$
$$\vec{B}(\vec{r},t) = B(\vec{r},t)\vec{e}_z$$

- Now we want to show that these equations satisfy Maxwell's Equations

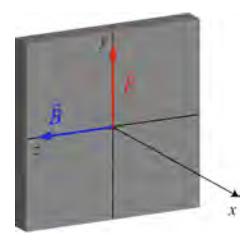
Gauss' Law for Electric Fields

- Let's start with Gauss' Law for electric fields
- For an electromagnetic wave, there is no enclosed charge ($q_{enc} = 0$), so we must show that our solution satisfies

$$\oint \vec{E} \bullet d\vec{A} = 0$$

- We can draw a rectangular Gaussian surface around a snapshot of the wave as shown to the right
- For the faces in y-z and x-y planes

$$\vec{E} \perp d\vec{A} \quad \Rightarrow \quad \vec{E} \cdot d\vec{A} = 0$$



- The faces in the x-z planes will contribute $+EA$ and $-EA$
- Thus the integral is zero and Gauss' Law for electric fields is satisfied

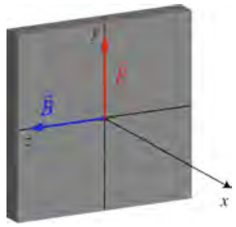
Gauss' Law for Magnetic Fields

- For Gauss' Law for magnetic fields we must show

$$\oint \vec{B} \bullet d\vec{A} = 0$$

- We use the same surface that we used for the electric field
- For the faces in the *y-z* and *x-z* planes

$$\vec{B} \perp d\vec{A} \quad \Rightarrow \quad \vec{B} \bullet d\vec{A} = 0$$



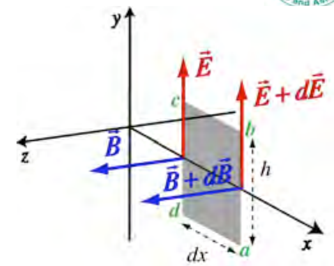
- The faces in the *x-y* planes will contribute +*BA* and -*BA*
- Thus our integral is zero and Gauss' Law for magnetic fields is satisfied

Faraday's Law

- Now let's tackle Faraday's Law

$$\oint \vec{E} \bullet d\vec{s} = -\frac{d\Phi_B}{dt}$$

- To evaluate the integral on the left side, we assume an integration loop in the x-y plane that has a width dx and height h as shown by the gray box in the figure
- The differential area of this rectangle is



$$d\vec{A} = \vec{n}dA = \vec{n}hdx \quad (\vec{n} \text{ in } +z \text{ direction})$$

- The electric and magnetic fields change as we move in the x direction

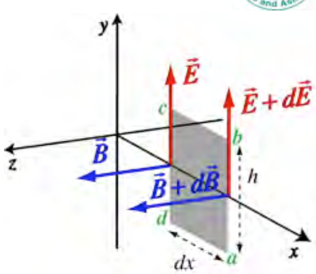
$$\vec{E}(x) \quad \Rightarrow \quad \vec{E}(x+dx) = \vec{E}(x) + d\vec{E}$$

Faraday's Law (2)

- The integral around the loop is

$$\oint \vec{E} \bullet d\vec{s} = (E + dE)h - Eh = dE \cdot h$$

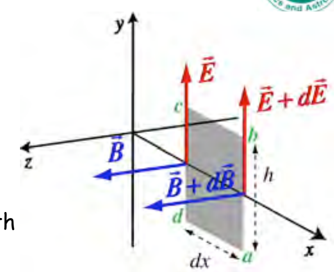
- We can split this integral over a closed loop into four pieces, integrating from
  - ***a*** to ***b***
  - ***b*** to ***c***
  - ***c*** to ***d***
  - ***d*** to ***a***



- The contributions to the integral parallel to the *x*-axis, integrating from ***b*** to ***c*** and from ***d*** to ***a***, are zero because the electric field is always perpendicular to the integration direction
- For the integrations along the *y*-direction, ***a*** to ***b*** and ***c*** to ***d***, one has the electric field parallel to the direction of the integration
  - The scalar product simply reduces to a conventional product

Faraday's Law (3)

- Because the electric field is independent of the *y*-coordinate, it can be taken out of the integration
  - The integral along each of the segments in the ±*y* direction is a simple product of the integrand, the electric field at the corresponding *y*-coordinate, times the length of the integration interval, *h*



- The right hand side is given by

$$-\frac{d\Phi_B}{dt} = -A\frac{dB}{dt} = -hdx\frac{dB}{dt}$$

- So we have

$$hdE = -hdx\frac{dB}{dt} \quad \Rightarrow \quad \frac{dE}{dx} = -\frac{dB}{dt}$$

Faraday's Law (4)

- The derivatives *dE*/*dx* and *dB*/*dt* are taken with respect to a single variable, although both *E* and *B* depend on both *x* and *t*
- Thus we can more appropriately write

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$$

- Taking our assumed form for the electric and magnetic fields we can execute the derivatives

$$\frac{\partial E}{\partial x} = \frac{\partial}{\partial x}\left(E_{\max}\sin\left(kx-\omega t\right)\right) = kE_{\max}\cos\left(kx-\omega t\right)$$

$$\frac{\partial B}{\partial t} = \frac{\partial}{\partial t}\left(B_{\max}\sin\left(kx-\omega t\right)\right) = -\omega B_{\max}\cos\left(kx-\omega t\right)$$

Faraday's Law (5)

- Which gives us

$$kE_{\max}\cos\left(kx-\omega t\right) = -\left(-\omega B_{\max}\cos\left(kx-\omega t\right)\right)$$

- We can relate the angular frequency ω and the angular wave number *k* as

$$\frac{\omega}{k} = \frac{2\pi f}{\left(\frac{2\pi}{\lambda}\right)} = f\lambda = c \quad (c \text{ is the speed of light})$$

- We can then write

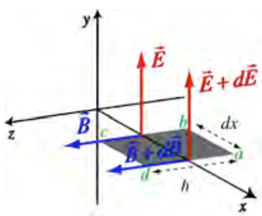
$$\frac{E_{\max}}{B_{\max}} = \frac{\omega}{k} = c \quad \Rightarrow \quad \frac{E}{B} = c$$

- Which implies that our assumed solution satisfies Faraday's Law as long as E/B = c

Ampere-Maxwell Law

- For electromagnetic waves there is no current

$$\oint \vec{B} \bullet d\vec{s} = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$$



- To evaluate the integral on the left hand side of this equation, we assume an integration loop in the plane that has a width I and height *h* depicted by the gray box in the figure
- The differential area of this rectangle is oriented along the +*y* direction
- The integral around the loop in a counter-clockwise direction (***a*** to ***b*** to ***c*** to ***d*** to ***a***) is given by

$$\oint \vec{B} \bullet d\vec{s} = -\left(B+dB\right)h + Bh = -dB \cdot h$$

- The parts of the loop that are parallel to the axis do not contribute

Ampere-Maxwell Law (2)

- The right hand side can be written as

$$\mu_0\varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0\varepsilon_0 \frac{dE \cdot A}{dt} = \mu_0\varepsilon_0 \frac{dE \cdot h \cdot dx}{dt}$$

- Substituting back into the Ampere-Maxwell relation we get

$$-dB \cdot h = \mu_0\varepsilon_0 \frac{dE \cdot h \cdot dx}{dt}$$

- Simplifying and expressing this equation in terms of partial derivatives as we did before we get

$$-\frac{\partial B}{\partial x} = \mu_0\varepsilon_0 \frac{\partial E}{\partial t}$$

- Putting in our assumed solutions gives us

$$-\left(kB_{\max}\cos\left(kx-\omega t\right)\right) = -\mu_0\varepsilon_0\omega E_{\max}\cos\left(kx-\omega t\right)$$

Ampere-Maxwell Law (3)

- We can rewrite the previous equation as

$$\frac{E_{\max}}{B_{\max}} = \frac{k}{\mu_0 \varepsilon_0 \omega} = \frac{1}{\mu_0 \varepsilon_0 c}$$

- This relationship also holds for the electric and magnetic field at any time

$$\frac{E}{B} = \frac{1}{\mu_0 \varepsilon_0 c}$$

- So our assumed solutions satisfy the Ampere Maxwell law if the ratio of *E*/*B* is

$$\frac{1}{\mu_0 \varepsilon_0 c}$$

The Speed of Light

- Our solutions for Maxwell's Equations were correct if

$$\frac{E}{B} = c \text{ and } \frac{E}{B} = \frac{1}{\mu_0 \varepsilon_0 c}$$

- We can see that

$$c = \frac{1}{\mu_0 \varepsilon_0 c} \Rightarrow c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

- Thus the speed of an electromagnetic wave can be expressed in terms of two fundamental constants related to electric fields and magnetic fields, the magnetic permeability and the electric permittivity of the vacuum
- If we put in the values of these constants that we have been using we get

$$c = \frac{1}{\sqrt{(1.26 \cdot 10^{-6} \text{ H/m})(8.85 \cdot 10^{-12} \text{ F/m})}} = 2.99 \cdot 10^{8} \text{ m/s}$$

The Speed of Light (2)

- This similarity implies that all electromagnetic waves travel at the speed of light
- Further this result implies that light is an electromagnetic wave
- The speed of light plays an important role in the theory of relativity
- The speed of light is always the same in any reference frame
- Thus if you send an electromagnetic wave out in a specific direction, any observer, regardless of whether that observer is moving toward you or away from you, will see that wave moving at the speed of ligh
- This amazing result leads to the theory of relativity
- The speed of light can be measured extremely precisely, much more precisely than we can determine the meter from the original reference standard. So now the speed of light is defined as precisely

$$c = 299,792,458 \text{ m/s}$$

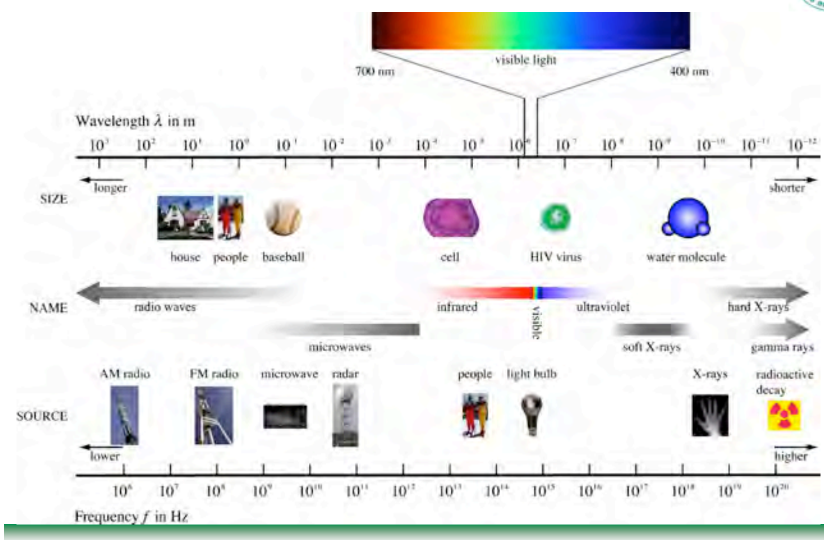
The Electromagnetic Spectrum

- All electromagnetic waves travel at the speed of light
- However, the wavelength and frequency of electromagnetic waves can vary dramatically
- The speed of light c, the wavelength λ, and the frequency f are related by

$$c = \lambda f$$

- Examples of electromagnetic waves include light, radio waves, microwaves, x-rays, and gamma rays
- This diverse spectrum is illustrated on the next slide

The Electromagnetic Spectrum (2)



Electromagnetic Spectrum (3)

- Electromagnetic waves exist with wavelengths ranging from 1000 m to less than 10^{-12} m and frequencies ranging from 10^{6} to 10^{12} Hz
- Certain ranges of wavelength and frequency have names that identify the most common application of those electromagnetic waves
- Visible light refers to electromagnetic waves ranging in wavelength from 400 nm to 700 nm
- The response of the human eye is peaked around 550 nm (green) and drops off quickly away from that wavelength
- Other wavelengths of electromagnetic waves are invisible to the human eye
- However, we can still detect the electromagnetic waves by other means. For example, we can feel electromagnetic waves in the infrared (wavelengths just longer than visible up to around m) as warmth.