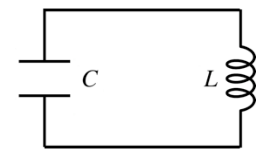
Physics for Scientists & Engineers 2

Spring Semester 2005

Lecture 29

Review

- Consider a circuit consisting of an inductor L and a capacitor C



- The charge on the capacitor as a function of time is given by

$$q = q_{max} \cos(\omega_0 t + \phi)$$

- The current in the inductor as a function of time is given by

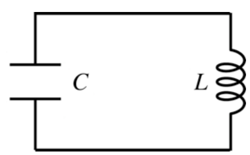
$$i = -i_{max} \sin(\omega_0 t + \phi)$$

- where ϕ is the phase and ω_0 is the angular frequency

$$\omega_0 = \sqrt{\frac{1}{LC}} = \frac{1}{\sqrt{LC}}$$

Review (2)

- The energy stored in the electric field of the capacitor *C* as a function of time is



$$U_E = \frac{q_{max}^2}{2C} \cos^2(\omega_0 t + \phi)$$

- The energy stored in the magnetic field of the inductor *L* as a function of time is

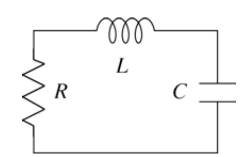
$$U_B = \frac{L}{2} i_{max}^2 \sin^2(\omega_0 t + \phi)$$

- The total energy stored in the circuit is given by

$$U = U_E + U_B = \frac{q_{max}^2}{2C}$$

RLC Circuit

- Now let's consider a single loop circuit that has a capacitor *C* and an inductance *L* with an added resistance *R*



- We observed that the energy of a circuit with a capacitor and and an inductor remains constant and that the energy translated from electric to magnetic and back gain with no losses
- If there is a resistance in the circuit, the current flow in the circuit will produce ohmic losses to heat
- Thus the energy of the circuit will decrease because of these losses

RLC Circuit (2)

- The rate of energy loss is given by

$$\frac{dU}{dt} = -i^2 R$$

- We can rewrite the change in energy of the circuit as a function time as

$$\frac{dU}{dt} = \frac{d}{dt}\left(U_E + U_B\right) = \frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} = -i^2 R$$

- Remembering that $i = dq/dt$ and $di/dt = d^2q/dt^2$ we can write

$$\frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} + i^2 R = \frac{q}{C}\frac{dq}{dt} + L\frac{dq}{dt}\frac{d^2q}{dt^2} + \left(\frac{dq}{dt}\right)^2 R = 0$$

RLC Circuit (3)

- We can then write the differential equation

$$L\frac{d^2q}{dt^2} + \frac{dq}{dt}R + \frac{q}{C} = 0$$

- The solution of this differential equation is

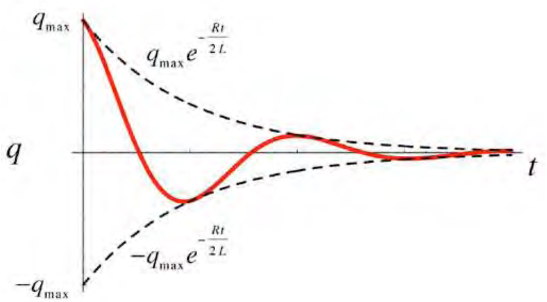
$$q = q_{\max} e^{-\frac{Rt}{2L}} \cos(\omega t + \phi)$$

- where

$$\omega = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

RLC Circuit (4)

- Now consider a single loop circuit that contains a capacitor, an inductor and a resistor
- If we charge the capacitor then hook it up to the circuit, we will observe a charge in the circuit that varies sinusoidally with time and while at the same time decreasing in amplitude
- This behavior with time is illustrated below



RLC Circuit (5)

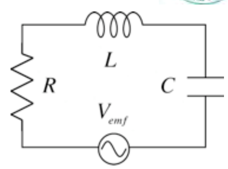
- The charge varies sinusoidally with but the amplitude is damped out with time
- After some time, no charge remains in the circuit
- We can study the energy in the circuit as a function of time by calculating the energy stored in the electric field of the capacitor

$$U_E = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}\frac{\left(q_{\max} e^{-\frac{Rt}{2L}} \cos(\omega t + \phi)\right)^2}{C} = \frac{q_{\max}^2}{2C} e^{-\frac{Rt}{L}} \cos^2(\omega t + \phi)$$

- We can see that the energy stored in the capacitor decreases exponentially and oscillates in time

Alternating Current

- Now we consider a single loop circuit containing a capacitor, an inductor, a resistor, and a source of emf



- This source of emf is capable producing a time varying voltage as opposed the sources of emf we have studied in previous chapters
- We will assume that this source of emf provides a sinusoidal voltage as a function of time given by

$$V_{emf} = V_{\max} \sin \omega t$$

- where ω is that angular frequency of the emf and $V_{\max}$ is the amplitude or maximum value of the emf

Alternating Current (2)

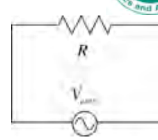
- The current induced in the circuit will also vary sinusoidally with time
- This time-varying current is called alternating current
- However, this current may not always remain in phase with the time-varying emf
- We can express the induced current as

$$i = I \sin(\omega t - \phi)$$

- where the angular frequency of the time-varying current is the same as the driving emf but the phase ϕ is not zero
- Note that traditionally the phase enters here with a negative sign
- Thus the voltage and the current in the circuit are not necessarily in phase

Circuit with Resistor

- To begin our analysis of *RLC* circuits, let's start with a circuit containing only a resistor and a source of time-varying emf as shown to the right



- Applying Kirchhof's loop rule to this circuit we get

$$V_{emf} - v_R = 0 \quad \Rightarrow \quad V_{emf} = v_R$$

- where v_R is the voltage drop across the resistor
- Substituting into our expression for the emf as a function of time we get

$$v_R = V_R \sin \omega t$$

- Remembering Ohm's Law, $V = iR$, we get

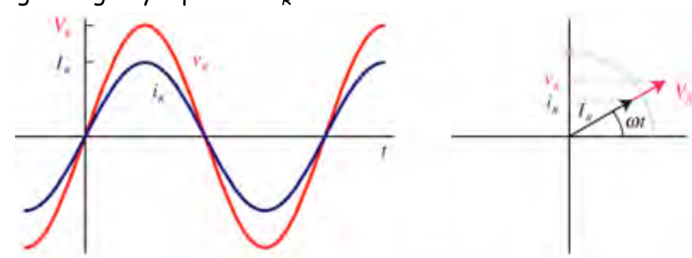
$$i_R = \frac{v_R}{R} = I_R \sin \omega t$$

Circuit with Resistor (2)

- Thus we can relate the current amplitude and the voltage amplitude by

$$V_R = I_R R$$

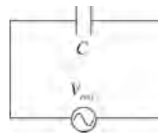
- We can represent the time varying current by a phasor I_R and the time-varying voltage by a phasor V_R as shown below



- The current flowing through the resistor and the voltage across the resistor are in phase, which means that the phase difference between the current and the voltage is zero

Circuit with Capacitor

- Now let's address a circuit that contains a capacitor and a time varying emf as shown to the right



- The voltage across the capacitor is given by Kirchhof's loop rule

$$v_C = V_C \sin \omega t$$

- Remembering that $q = CV$ for a capacitor we can write

$$q = Cv_C = CV_C \sin \omega t$$

- We would like to know the current as a function of time rather than the charge so we can write

$$i_C = \frac{dq}{dt} = \frac{d\left(CV_C \sin \omega t\right)}{dt} = \omega CV_C \cos \omega t$$

Circuit with Capacitor (2)

- We can rewrite the last equation by defining a quantity that is similar to resistance and is called the capacitive reactance

$$X_C = \frac{1}{\omega C}$$

- Which allows us to write

$$i_C = \frac{V_C}{X_C} \cos \omega t$$

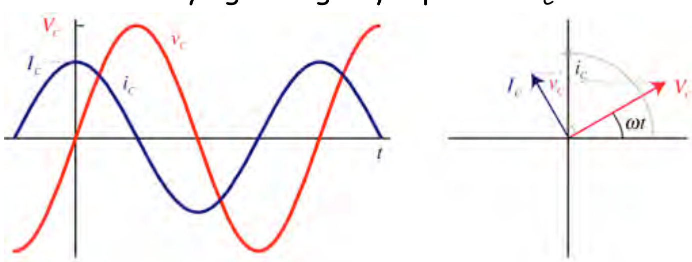
- We can now express the current in the circuit as

$$i_C = I_C \cos \omega t = I_C \sin\left(\omega t + 90°\right)$$

- We can see that the current and the time varying emf are out of phase by 90°

Circuit with Capacitor (3)

- We can represent the time varying current by a phasor I_C and the time-varying voltage by a phasor V_C as shown below



- The current flowing this circuit with only a capacitor is similar to the expression for the current flowing in a circuit with only a resistor except that the current is out of phase with the emf by 90°

Circuit with Capacitor (4)

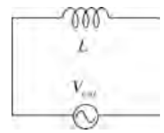
- We can also see that the amplitude of voltage across the capacitor and the amplitude of current in the capacitor are related by

$$V_C = I_C X_C$$

- This equation resembles Ohm's Law with the capacitive reactance replacing the resistance
- One major difference between the capacitive reactance and the resistance is that the capacitive reactance depends on the angular frequency of the time-varying emf

Circuit with Inductor

- Now let's consider a circuit with a source of time-varying emf and an inductor as shown to the right



- We can again apply Kirchhof's Loop Rule to this circuit to obtain the voltage across the inductor as

$$v_L = V_L \sin\omega t$$

- A changing current in an inductor will induce an emf given by

$$v_L = L\frac{di_L}{dt}$$

- So we can write

$$L\frac{di_L}{dt} = V_L \sin\omega t \quad \Rightarrow \quad \frac{di_L}{dt} = \frac{V_L}{L}\sin\omega t$$

Circuit with Inductor (2)

- We are interested in the current rather than its time derivative so we integrate

$$i_L = \int \frac{di_L}{dt}dt = \int \frac{V_L}{L}\sin\omega t dt = -\frac{V_L}{\omega L}\cos\omega t$$

- We define inductive reactance as

$$X_L = \omega L$$

- which, like the capacitive reactance, is similar to a resistance
- We can then write

$$v_L = i_L X_L$$

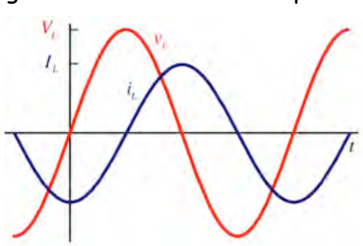
- which again resembles Ohm's Law except that the inductive reactance depends on the angular frequency of the time-varying emf

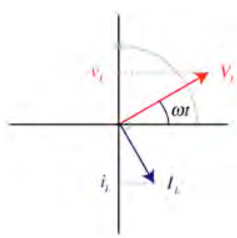
Circuit with Inductor (3)

- The current in the inductor can then be written as

$$i_L = -\frac{v_L}{X_L}\cos\omega t = -I_L\cos\omega t = I_L\sin(\omega t - 90°)$$

- Thus the current flowing in a circuit with an inductor and a source time-varying emf will be -90° out of phase with the emf





- We can write the relationship between the amplitude of the current and the amplitude of the voltage as

$$V_L = I_L X_L$$