Physics for Scientists & Engineers 2

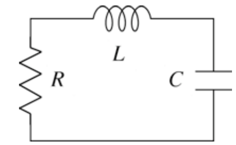
Spring Semester 2005

Lecture 30

Review

- If we have a single loop RLC circuit, the charge in the circuit as a function of time is given by

$$q = q_{\max} e^{-\frac{Rt}{2L}} \cos(\omega t + \phi)$$



- Where

$$\omega = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

- The energy stored in the capacitor as a function of time is given by

$$U_E = \frac{q_{\max}^2}{2C} e^{-\frac{Rt}{L}} \cos^2(\omega t + \phi)$$

Review (2)

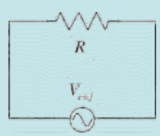
- Time-varying emf

$$V_{emf} = V_{\max} \sin \omega t$$

- Time-varying emf V_R with resistor

$$i_R = \frac{V_R}{R} = I_R \sin \omega t$$

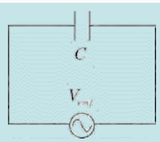
Resistance



- Time-varying emf V_C with capacitor

$$X_C = \frac{1}{\omega C} \qquad i_C = \frac{V_C}{X_C} \sin(\omega t + 90°)$$

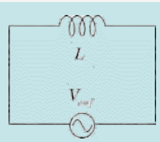
Capacitive Reactance



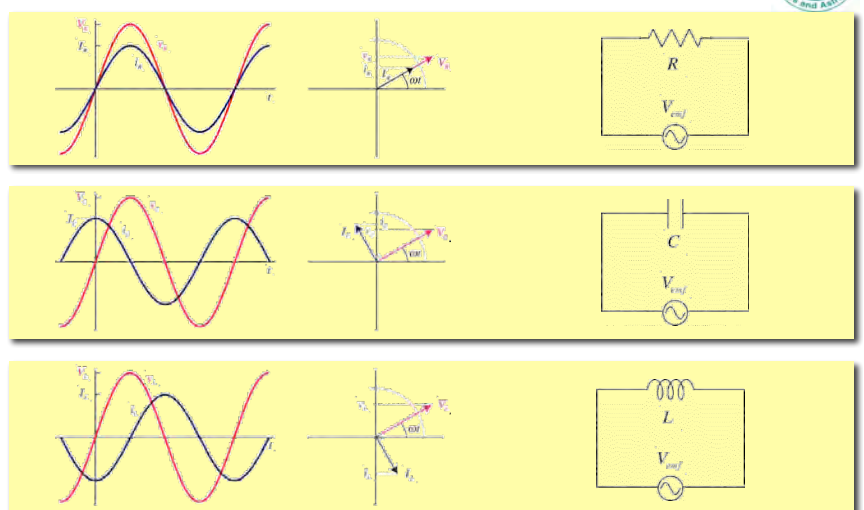
- Time-varying emf V_L with inductor

$$X_L = \omega L \qquad i_L = \frac{V_L}{X_L} \sin(\omega t - 90°)$$

Inductive Reactance

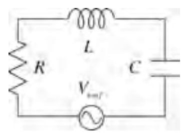


Review (3)

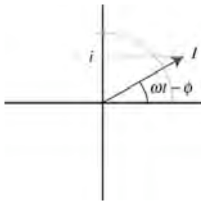


Series RLC Circuit

- Consider a single loop circuit that has a resistor, a capacitor, an inductor, and a source of time-varying emf



- We can describe the time-varying currents in these circuit elements using a phasor I
- The projection of I on the vertical axis represents the current flowing in the circuit as a function of time
  - The angle of the phasor is given by ωt - ϕ



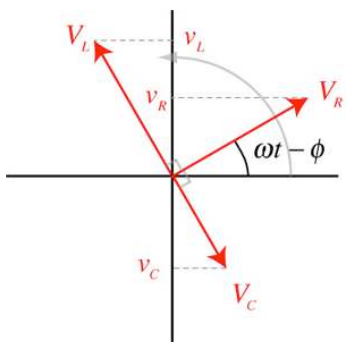
- We can also describe the voltage in terms of a phasor V
- The time-varying currents and voltages in the circuit can have different phases

Series RLC Circuit (2)

- We can describe the current flowing in the circuit and the voltage across the various components
  - Resistor
    - The voltage v_R and current i_R are in phase with each other and the voltage phasor v_R is in phase with the current phasor I
  - Capacitor
    - The current i_C leads the voltage v_C by 90° so that the voltage phasor v_C will have an angle 90° less than I and v_R
  - Inductor
    - The current i_L lags behind the voltage v_L by so that voltage phasor v_L will have an angle 90° greater than I and v_R

Series RLC Circuit (3)

- The voltage phasors for an RLC circuit are shown below



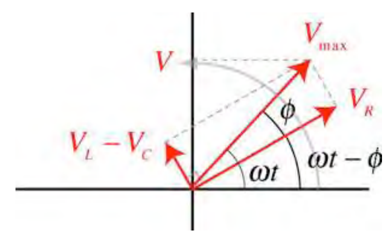
- The instantaneous voltages across each of the components are represented by the projections of the respective phasors on the vertical axis

Series RLC Circuit (4)

- Kirchhof's loop rules tells that the voltage drops across all the devices at any given time in the circuit must sum to zero, which gives us

$$V - v_R - v_C - v_L = 0 \quad \Rightarrow \quad V = v_R + v_C + v_L$$

- The voltage can be thought of as the projection of the vertical axis of the phasor V_{max} representing the time-varying emf in the circuit as shown below



- In this figure we have replaced the sum of the two phasors V_L and V_C with the phasor V_L - V_C

Series RLC Circuit (5)

- The sum of the two phasors V_L - V_C and V_R must equal V_{max} so
  $$V_{max}^2 = V_R^2 + \left(V_L - V_C\right)^2$$
- Now we can put in our expression for the voltage across the components in terms of the current and resistance or reactance
  $$V_{max}^2 = \left(IR\right)^2 + \left(IX_L - IX_C\right)^2$$
- We can then solve for the current in the circuit
  $$I = \frac{V_{max}}{\sqrt{R^2 + \left(X_L - X_C\right)^2}}$$
- The denominator in the equation is called the impedance
  $$Z = \sqrt{R^2 + \left(X_L - X_C\right)^2}$$
- The impedance of a circuit depends on the frequency of the time-varying emf

Series RLC Circuit (6)

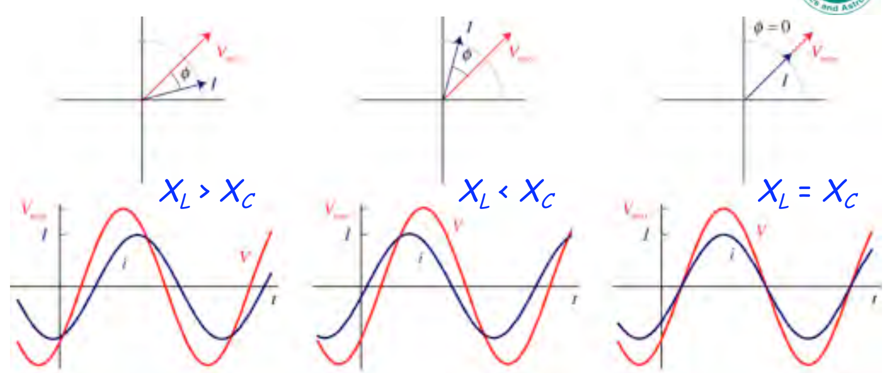
- The current flowing in an alternating current circuit depends on the difference between the inductive reactance and the capacitive reactance
- We can express the difference between the inductive reactance and the capacitive reactance in terms of the phase constant ϕ
- This phase constant is defined as the phase difference between voltage phasors V_R and V_L - V_C

$$\phi = \tan^{-1}\left(\frac{V_L - V_C}{V_R}\right) = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

Series RLC Circuit (7)

- Thus we have three conditions for an alternating current circuit
  - For $X_L > X_C$, ϕ is positive, and the current in the circuit will lag behind the voltage in the circuit
    - This circuit will be similar to a circuit with only an inductor, except that the phase constant is not necessarily 90°
  - For $X_L < X_C$, ϕ is negative, and the current in the circuit will lead the voltage in the circuit
    - This circuit will be similar to a circuit with only a capacitor, except that the phase constant is not necessarily -90°
  - For $X_L = X_C$, ϕ is zero, and the current in the circuit will be in phase with the voltage in the circuit
    - This circuit is similar to a circuit with only a resistance
    - When ϕ = 0 we say that the circuit is in resonance

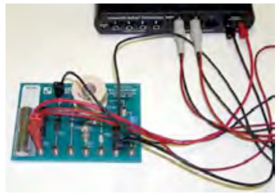
Series RLC Circuit (8)

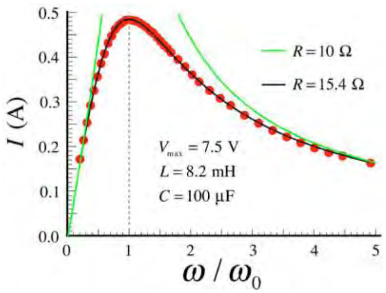


- For $X_L = X_C$ and $\phi = 0$ we get the maximum current in the circuit and we can define a resonant frequency
  $$\omega L - \frac{1}{\omega C} = 0 \quad \Rightarrow \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

Real-life RLC Circuit (2)

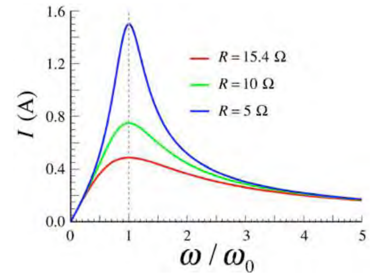
- Let's study a real-life circuit
  - R = 10 Ω
  - L = 8.2 mH
  - C = 100 μF
  - V_{max} = 7.5 V
- We measure the current in the circuit as a function of the frequency of the time-varying emf
- We see the correct resonant frequency (peak at ω/ω_0 = 1)
  - L and C must be accurate
- However, our formula for the current (green line) using R = 10 Ω does not agree with the measurements
  - We must use R = 15.4 Ω
    - The inductor has a resistance even at resonance





Resonant Behavior of RLC Circuit

- The resonant behavior of an RLC circuit resembles the response of a damped oscillator
- Here we show the calculated maximum current as a function of the ratio of the angular frequency of the time varying emf divided by the resonant angular frequency, for a circuit with V_{max} = 7.5 V, L = 8.2 mH, C = 100 μF, and three resistances
- One can see that as the resistance is lowered, the maximum current at the resonant angular frequency increases and there is a more pronounced resonant peak



Energy and Power in RLC Circuits

- When an RLC circuit is in operation, some of the energy in the circuit is stored in the electric field of the capacitor, some of the energy is stored in the magnetic field of the inductor, and some energy is dissipated in the form of heat in the resistor
- The energy stored in the capacitor and inductor do not change in steady state operation
- Therefore the energy transferred from the source of emf to the circuit is transferred to the resistor
- The rate at which energy is dissipated in the resistor is the power P given by

$$P = i^2 R = \left(I \sin(\omega t - \phi)\right)^2 R = I^2 R \sin^2(\omega t - \phi)$$

- The average power is given by

$$\langle P \rangle = \frac{1}{2} I^2 R = \left(\frac{I}{\sqrt{2}}\right)^2 R$$

Energy and Power (2)

- We define the root-mean-square (rms) current to be $I_{rms} = \frac{I}{\sqrt{2}}$
- So we can write the average power as $\langle P \rangle = I_{rms}^2 R$
- We can make similar definitions for other time-varying quantities
  - rms voltage: $V_{rms} = \frac{V}{\sqrt{2}}$
  - rms time-varying emf: $V_{\max,rms} = \frac{V_{\max}}{\sqrt{2}}$
- The currents and voltages measured by a alternating current ammeter or voltmeter are rms values
- For example, we normally say that the voltage in the wall socket is 110 V
- This rms voltage would correspond to a maximum voltage of $\sqrt{2} \cdot 110\ \text{V} \approx 156\ \text{V}$

Energy and Power (3)

- We can then re-write our formula for the current as

$$I_{rms} = \frac{V_{\max,rms}}{Z} = \frac{V_{\max,rms}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

- Which allows us to express the average power dissipated as

$$\langle P \rangle = I_{rms}^2 R = \frac{V_{\max,rms}}{Z} I_{rms} R = I_{rms} V_{\max,rms} \frac{R}{Z}$$

- We can relate the phase constant to the ratio of the maximum value of the voltage across the resistor divided by the maximum value of the time-varying emf

$$\cos\phi = \frac{V_R}{V_{\max}} = \frac{IR}{IZ} = \frac{R}{Z} \quad \Rightarrow \quad \langle P \rangle = I_{rms} V_{rms} \cos\phi$$

- We can see that the maximum power is dissipated when $\phi = 0$
- We call cos(ϕ) the power factor

Transformers

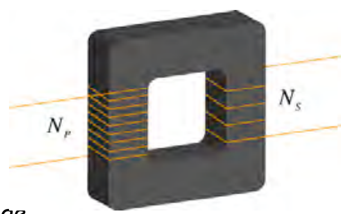
- When using or generating electrical power, high currents and low voltages are desirable for convenience and safety
- When transmitting electric power, high voltages and low currents are desirable
  - The power loss in the transmission wires goes as $P = I^2 R$
  - Assume we have 500 MW of power to transmit
  - If we transmit at 750 kV, the current would be 667 A
  - If the resistance of the power lines is 200 Ω, the power dissipated in the power lines is 89 MW
    - 18% loss
  - Suppose we transmit at 375 kV instead
    - 75% loss
- The ability to raise and lower alternating voltages is useful in everyday life

Transformers (2)

- To transform alternating currents and voltages from high to low one uses a transformer
- A transformer that takes voltages from lower to higher is called a step-up transformer and a transformer that takes voltages from higher to lower is called a step-down transformer
- A transformer consists of two sets of coils wrapped around an iron core as illustrated
- Consider the primary windings with N_P turns connected to a source of emf

$$V_{emf} = V_{\max} \sin \omega t$$

- We can assume that the primary windings act as an inductor
- The current is out of phase with the voltage and no power is delivered to the transformer



Transformers (3)

- Now consider the second coil with N_S turns
- The time-varying emf in the primary coil induces a time-varying magnetic field in the iron core
- This core passes through the secondary coil
- Thus a time-varying voltage is induced in the secondary coil described by Faraday's Law

$$V_{emf} = -N \frac{d\Phi_B}{dt}$$

- Because both the primary and secondary coils experience the same changing magnetic field we can write

$$\frac{V_P}{N_P} = \frac{V_S}{N_S} \quad \Rightarrow \quad V_S = V_P \frac{N_S}{N_P}$$

Transformers (4)

- If we now connect a resistor R across the secondary windings, a current will begin to flow through the secondary coil
- The power in the secondary circuit is then $P_S = I_S V_S$
- This current will induce a time-varying magnetic field that will induce an emf in the primary coil
- The emf source then will produce enough current I_P to maintain the original emf
- This induced current will not be 90° out of phase with the emf thus power can be transmitted to the transformer
- Energy conservation tells that the power produced by the emf source in the primary coil will be transferred to the secondary coil so we can write

$$P_P = I_P V_P = P_S = I_S V_S \quad \Rightarrow \quad I_S = I_P \frac{V_P}{V_S} = I_P \frac{N_P}{N_S}$$

Transformers (5)

- When the secondary circuit begins to draw current, current must be supplied to the primary circuit
- We can define the current in the secondary circuit as $V_S = I_S R$
- We can then write the primary current as\

$$I_P = \frac{N_S}{N_P} I_S = \frac{N_S}{N_P}\frac{V_S}{R} = \frac{N_S}{N_P}\left(V_P \frac{N_S}{N_P}\right)\frac{1}{R} = \left(\frac{N_S}{N_P}\right)^2 \frac{V_P}{R}$$

- With an effective primary resistance of

$$R_P = \frac{V_P}{I_P} = V_P \left(\frac{N_P}{N_S}\right)^2 \frac{R}{V_P} = \left(\frac{N_P}{N_S}\right)^2 R$$

- Note that these equations assume no losses in the transformers and that the load is purely resistive
  - Real transformers have small losses
- Another application of transformers is impedance matching
  - Stereo amplifier to stereo speakers