Physics for Scientists & Engineers 2

Spring Semester 2005

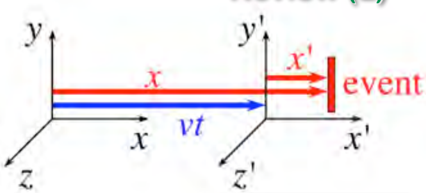
Lecture 49

Review

- Time Dilation $\Delta t = \dfrac{\Delta t_0}{\sqrt{1-(v/c)^2}} = \gamma \Delta t_0$
- Length Contraction $L = \dfrac{c\Delta t_0}{\gamma} = \dfrac{L_0}{\gamma}$
- Frequency, Wavelength Shift $f = f_0\sqrt{\dfrac{c \mp v}{c \pm v}}$ $\lambda = \lambda_0\sqrt{\dfrac{c \pm v}{c \mp v}}$
- Red-shift $z = \dfrac{\Delta\lambda}{\lambda_0} = \dfrac{\lambda - \lambda_0}{\lambda_0} = \sqrt{\dfrac{c \pm v}{c \mp v}} - 1$

  (upper signs for moving away from each other, lower for towards each other)

Review (2)



Note:

Lorentz transformation contains both time dilation and length contraction

Galilei	*Lorentz*	*Inverse*
$x' = x - vt$	$x' = \gamma(x - vt)$	$x = \gamma(x' + vt)$
$y' = y$	$y' = y$	$y = y'$
$z' = z$	$z' = z$	$z = z'$
$t' = t$	$t' = \gamma(t - vx/c^2)$	$t = \gamma(t' + vx'/c^2)$

Small v: Galilei special case of Lorentz

Review (3)

- Relativistic addition: $v_{1+2} = \dfrac{v_1 + v_2}{1 + \dfrac{v_1 v_2}{c^2}}$
- Relativistic momentum $\vec{p} = \gamma\, m\vec{v}$

Kinetic Energy

- Non-relativistic $K = \frac{1}{2}mv^2$
- Relativistic: need to consider the contribution of the mass to the energy
- Energy contained in the mass of a particle

$$E_0 = mc^2$$

- This is the energy that a particle has when it is at rest; sometimes also called "rest energy"
- When a particle is in motion, then its energy increases, just like the time becomes dilated:

$$E = \gamma\, E_0 = \gamma\, mc^2$$

Kinetic Energy (2)

- From the energy, we can obtain the kinetic energy by subtraction of the rest energy:

$$K = E - E_0 = (\gamma - 1)E_0 = (\gamma - 1)mc^2$$

- Non-relativistic limit (v<<c):

$$(1 - x^2)^{-1/2} = 1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4 + \ldots$$
$$\gamma = 1 + \tfrac{1}{2}\beta^2 = 1 + \tfrac{1}{2}v^2 / c^2$$

$$K = (\gamma - 1)mc^2$$
$$= ((1 + \tfrac{1}{2}v^2 / c^2) - 1)mc^2$$
$$= \tfrac{1}{2}mv^2$$

Momentum and Energy

- Start with the results for momentum and energy and square:

$$\vec{p} = \gamma\, m\vec{v} \Rightarrow p^2 = \gamma^2 m^2 v^2$$
$$E = \gamma\, mc^2 \Rightarrow E^2 = \gamma^2 m^2 c^4$$

- Square of the gamma-factor:

$$\gamma^2 = \frac{1}{1-\beta^2} = \frac{1}{1 - v^2/c^2} = \frac{c^2}{c^2 - v^2}$$

- Express E^2 as a function of p^2:

$$E^2 = \frac{c^2}{c^2 - v^2}m^2c^4 = \frac{c^2 - v^2 + v^2}{c^2 - v^2}m^2c^4 = m^2c^4 + \frac{v^2}{c^2 - v^2}m^2c^4$$
$$= m^2c^4 + \frac{c^2}{c^2 - v^2}m^2v^2c^2 = m^2c^4 + \gamma^2 m^2 v^2 c^2$$
$$= m^2c^4 + p^2c^2$$

Momentum and Energy (2)

- Finally we take square root

$$E = \sqrt{p^2c^2 + m^2c^4}$$

- Negative root: antimatter (Dirac)
- Case of zero mass (photons):

$$E = pc$$

Speed, Energy & Momentum

- Divide momentum by energy

$$\frac{p}{E}=\frac{\gamma\ mv}{\gamma\ mc^2}=\frac{v}{c^2}\Rightarrow v=\frac{pc^2}{E}$$

- So we can express beta-factor as:

$$\beta=\frac{v}{c}=\frac{pc}{E}$$

- This gives us an additional energy-momentum relationship:

$$\beta=\frac{pc}{E}\Rightarrow p=\frac{\beta E}{c}\ \text{or}\ E=\frac{pc}{\beta}$$

Example: Energy&Momentum

Question:

- If an electron has a speed of 99% of that of light, what is its total energy, what is its kinetic energy, and what is its momentum?

Answer:

- Rest energy of electron $E_0 = 5.11\cdot 10^5\ \text{eV} = 0.511\ \text{MeV}$
- Rest mass $m = 0.511\ \text{MeV}/c^2$
  (remember 1 eV = $1.609\cdot 10^{-19}$ J)
- Gamma for 99% of c: $$\gamma=\frac{1}{\sqrt{1-\beta^2}}=\frac{1}{\sqrt{1-0.99^2}}=7.09$$
- Total energy: $E=\gamma\ E_0 = 7.09\cdot 0.511\ \text{MeV} = 3.62\ \text{MeV}$

Example: Energy&Momentum (2)

Answer: (cont.)

- Kinetic energy of electron in this case:

$$K=(\gamma-1)E_0 = 6.09\cdot 0.511\ \text{MeV} = 3.11\ \text{MeV}$$

- Momentum: $p=\frac{\beta E}{c}=\frac{0.99\cdot 3.62\ \text{MeV}}{c}=3.58\ \text{MeV}/c$
- An accelerator to achieve this is quite small and inexpensive; but in the last 1% is where it gets big and expensive
- At SLAC, the speed reached by electrons is 99.9999999% of the speed of light, and it takes an accelerator of length 3 km to accomplish this.

Mass and Gravity

- Newtonian theory of gravity

$$ma = F_g = \frac{GmM}{r^2}$$

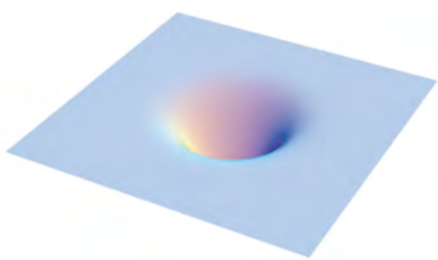
- Mass, m, appears on both sides of this equation
  - Right side: source of the interaction (gravitational mass)
  - Left side: mass undergoing acceleration (inertial mass)
  - Experimental finding: gravitational mass = inertial mass
- Newtonian gravity works extremely well, except near very large masses
  - Example: Small deviations from observed orbit of Mercury

Equivalence Principle

- Einstein (1907): *"If a person falls freely he will not feel his own weight."*
- Consequence: you cannot distinguish if you are in an accelerating reference frame, or if you are subject to the gravitational force.
- Equivalence Principle: **"All local freely falling non-rotating laboratories are fully equivalent for the performance of all physical experiments"**
- Space and time are locally curved due to the presence of masses

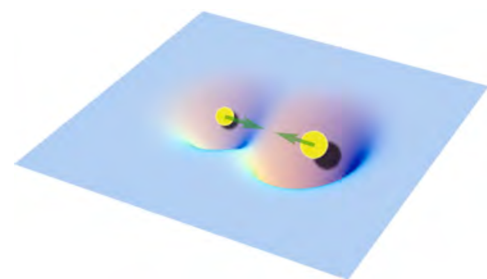
Curved Space

- Two-dimensional example: Imagine a flat rubber sheet, suspended at the edges. Put a bowling ball on it: local deformation



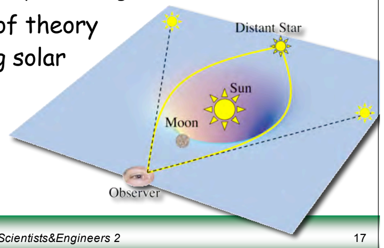
Curved Space (2)

- Now put two objects on the rubber sheet
- Objects are attracted to each other and move towards each other => Gravitational interaction



Curved Space (3)

- Objects do not move on Euclidean straight lines, but on lines that correspond to the shortest distance in curved space time
- Even though light does not have mass, it also moves on the shortest path through curved space-time
  - => Light gets deflected (slightly) near large masses
- Spectacular confirmation of theory of general relativity during solar eclipse in 1919
  - Angular deflection of 1.75" predicted and observed
  - "Gravitational lensing"



Gravitational Lensing

Arcs: gravitational lensing due to massive objects in line of sight.

Galaxy Cluster Abell 2218 (Andrew Fruchter (STScI) et al., WFPC2, HST, NASA)

Schwarzschild Radius

- Definition: $R_s = \frac{2GM}{c^2}$

  (depends only on the mass, M)
- Example: Earth

$$R_{s,E} = \frac{2GM_E}{c^2} = \frac{2\cdot(6.67\cdot10^{-11}\ \mathrm{Nm^2/kg^2})\cdot(6.0\cdot10^{24}\,\mathrm{kg})}{(3.0\cdot10^8\ \mathrm{m/s})^2} = 8.9\ \mathrm{mm}$$

- If the Schwarzschild radius is bigger than the physical radius of an object, then we call this object a black hole
- Stars with $M > 15\ M_{\mathrm{sun}}$ collapse into a black hole at the end of their live

GPS - Relativity Works!

- GPS-system consists of 24 satellites with atomic clocks on board
- Typically, a receiver gets timing signals from at least 4 satellites simultaneously
- From the timing of the signals, receiver can determine its position

$$|\vec{r}_r - \vec{r}_i| = c\cdot|t_r - t_i| \quad \text{for } i = 1,\dots,4$$

- Time dilation effects need to be corrected for, because satellites move with 4 km/s rel. to Earth.