Newtonian Mechanics

1   HISTORY

Isaac Newton solved the premier scientific problem of his day, which was to explain the motion of the planets. He published his theory in the famous book known as *Principia*. The full Latin title of the book[1] may be translated into English as *Mathematical Principles of Natural Philosophy*.

The theory that the planets (including Earth) revolve around the sun was published by Nicolaus Copernicus in 1543. This was a revolutionary idea! The picture of the Universe that had been developed by astronomers before Copernicus had the Earth at rest at the center, and the sun, moon, planets and stars revolving around the Earth. But this picture failed to explain accurately the observed planetary positions. The failure of the Earth-centered theory led Copernicus to consider the sun as the center of planetary orbits. Later observations verified the Copernican theory. The important advances in astronomical observations were made by Galileo and Kepler.

Galileo Galilei was perhaps the most remarkable individual in the history of science. His experiments and ideas changed both physics and astronomy. In physics he showed that the ancient theories of Aristotle, which were still accepted in Galileo's time, are incorrect. In astronomy he verified the Copernican picture of the Universe by making the first astronomical observations with a telescope.

Galileo did not invent the telescope but he made some of the earliest telescopes, and his telescopes were the best in the world at that time. Therefore he discovered many things about the the solar system and stars:

- craters and mountains on the moon

- the moons of Jupiter

- the phases of Venus

- the motion of sunspots

---

[1]Philosophiae Naturalis Principia Mathematica

- the existence of many faint stars

These discoveries provided overwhelming evidence in favor of the Copernican model of the solar system.

Johannes Kepler had extensive data on planetary positions, as functions of time, from observations collected earlier by Tycho Brahe. He analyzed the data based on the Copernican model, and deduced three empirical laws of planetary motion:

Kepler's Laws

1. The planets move on elliptical orbits with the sun at one focal point.

2. The radial vector sweeps out equal areas in equal times.

3. The square of the period of revolution is proportional to the cube of the semimajor axis of the ellipse.

Newton started with the results of Galileo and Kepler. His goal, then, was to explain why. *Why do the planets revolve around the sun in the manner discovered by Galileo and Kepler?* In particular, what is the explanation for the *mathematical regularities* in Kepler's laws of orbital motion? To answer this question, Newton had to develop the laws of motion and the law of universal gravitation. And, to analyze the motion he invented a new branch of mathematics, which we now call *Calculus*.

The solution to planetary motion was published in *Principia* in 1687. Newton had solved the problem some years earlier, but kept it secret. He was visited in 1684 by the astronomer Edmund Halley. Halley asked what force would keep the planets in elliptical orbits. Newton replied that the force must be an inverse-square law, which he had proven by mathematical analysis; but he could not find the paper on which he had written the calculations! After further correspondence, Halley realized that Newton had made great advances in physics but had not published the results. With Halley's help, Newton published *Principia* in which he explained his theories of motion, gravity, and the solar system.

After the publication of *Principia*, Newton was the most reknowned scientist in the world. His achievement was fully recognized during his lifetime. Today scientists and engineers still use Newton's theory of mechanics. Classical mechanics breaks down at atomic dimensions, but for engineering systems Newton's theory is valid and extremely accurate.

How is this early history of science relevant to the study of calculus? Newton used calculus for analyzing motion, although he published the calculations in *Principia* using older methods of geometrical analysis. (He feared that the new mathematics—calculus—would not be understood or accepted.)

Ever since that time, calculus has been necessary to the understanding of physics and its applications in engineering and science.

In this chapter we'll study some basic applications of calculus to classical mechanics.

2 POSITION, VELOCITY, AND ACCELERATION

2.1 Position and velocity

Suppose an object M moves along a straight line.[2] We describe its motion
by giving the position x as a function of time t, as illustrated in Fig. 1. The
variable x is the *coordinate*, i.e., the displacement from a fixed point 0 called
the *origin*. Physically, the line on which M moves might be pictured as a
road, or a track. Mathematically, the positions form a representation of the
ideal real line. The coordinate x is positive if M is to the right of 0, or
negative if to the left. The absolute value $|x|$ is the distance from 0. The
possible positions of M are in one-to-one correspondence with the set of real
numbers. Hence *position* is a continuous function $x(t)$ of the independent
variable t, time.

Example 1. What is the position as a function of time if M is at rest at a
point 5 m to the left of the origin?

Solution. Because M is not moving, the function $x(t)$ is just a constant,

$$x(t) = -5\,\text{m}. \tag{1}$$

Note that the position has both a number (-5) and a unit (m, for meter).
In this case the number is negative, indicating a position to the left of 0.
The number alone is not enough information. The unit is required. The unit
may be changed by multiplying by a conversion factor. For example, the
conversion from meters (m) to inches (in) is

$$5\,\text{m} = 5\,\text{m} \times \frac{38\,\text{in}}{1\,\text{m}} = 190\,\text{in}. \tag{2}$$

(There are 38 inches per meter.) So, the position could just as well be written
as

$$x(t) = -190\,\text{in}. \tag{3}$$

Equations (3) and (1) are equivalent. This example shows why the number
alone is not enough: The number depends on the unit.

A Comment on Units. In physical calculations it is important to keep
track of the units—treating them as algebraic quantities. Dropping the unit
will often lead to a failed calculation. Keeping the units has a bonus. It is a
method of error checking. If the final unit is not correct, then there must be
an error in the calculation; we can go back and figure out how to correct the

---

[2]We'll call the moving object M. The letter M could stand for "moving" or "mass."

calculation.

Example 2. A car travels on a straight road, toward the East, at a constant speed of 35 mph. Write the position as a function of time. Where is the car after 5 minutes?

Solution. The origin is not specified in the statement of the problem, so let's say that $x = 0$ at time $t = 0$. Then the position as a function of time is

$$x(t) = + \left(35 \frac{\text{mi}}{\text{hr}} \right) t, \tag{4}$$

where positive x is east of the origin. Equation (4) is based on the formula

distance = speed × time,

familiar from grade school; or, taking account of the signs,

displacement = velocity × time.

Any distance must be positive. 'Displacement' may be positive or negative. Similarly, 'speed' must be positive, but 'velocity' may be positive or negative, negative meaning that M is moving to smaller x.

After 5 minutes, the position is

$$
\begin{aligned}
x(5\,\text{min}) &= 35 \frac{\text{mi}}{\text{hr}} \times 5\,\text{min} \\
&= 35 \frac{\text{mi}}{\text{hr}} \times 5\,\text{min} \times \frac{1\,\text{hr}}{60\,\text{min}} \\
&= 2.917\,\text{mi}.
\end{aligned}
\tag{5}
$$

(We multiply by the conversion factor, $1\,\text{hr}/60\,\text{min}$, to reduce the units.) The position could be expressed in feet, as

$$x(5\,\text{min}) = 2.917\,\text{mi} \times \frac{5280\,\text{ft}}{1\,\text{mi}} = 15400\,\text{ft}. \tag{6}$$

So, after 5 minutes the car is 15400 feet east of its initial position.

$$\star \qquad \star \qquad \star$$

It is convenient to record some general, i.e., abstract formulas. If M is at rest at x_0, then the function $x(t)$ is

$$x(t) = x_0. \qquad \text{(object at rest)} \tag{7}$$

If M moves with constant velocity v_0 then

$$x(t) = x_0 + v_0 t. \qquad \text{(object with constant velocity)} \tag{8}$$

Figure 2 illustrates graphs of these functions. The abscissa (horizontal axis) is the independent variable t, and the ordinate (vertical axis) is the dependent variable x. The slope in the second graph is v_0. Note that the units of v_0 must be a length unit divided by a time unit, because

$$v_0 = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\Delta x}{\Delta t}; \tag{9}$$

for example, the units could be m/s. But the slope in a graph is the derivative of the function! Thus, *the derivative of the position $x(t)$ is the velocity $v(t)$.*

So far we have considered only constant velocity. If the velocity is not constant, then the *instantaneous velocity* at a time t is the slope of the curve of x versus t, i.e., the slope of the tangent line. This is nothing but the derivative of $x(t)$. Letting $v(t)$ denote the velocity,

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}. \tag{10}$$

Another notation for the time-derivative, often used in mechanics, is $\dot{x}(t) \equiv dx/dt$. Because $v(t)$ is defined by the limit $\Delta t \to 0$, there is an instantaneous velocity at every t. We summarize the analysis by a definition:

Definition. The velocity $v(t)$ is a function of time t, defined by
$v(t) = dx/dt$.

2.2   Acceleration

If the velocity is changing then the object M is accelerating. The *acceleration* is defined as the time-derivative of the velocity,

$$a(t) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}. \qquad \text{(definition of acceleration)} \tag{11}$$

Taking the limit $\Delta t \to 0$, the acceleration is defined at every instant. Also, because $v = dx/dt$, the acceleration a is the *second derivative* of $x(t)$,

$$a(t) = \frac{d^2x}{dt^2} = \ddot{x}(t). \tag{12}$$

Example 3.   A car accelerates away from a stop sign, starting at rest. Assume the acceleration is a constant $5\,\text{m/s}^2$ for 3 seconds, and thereafter is 0. (a) What is the final velocity of the car? (b) How far does the car travel from the stop sign in 10 seconds?

Solution.   (a) Let the origin be the stop sign. During the 3 seconds while the car is accelerating, the acceleration is constant and so the velocity function

must be

$$v(t) = at, \tag{13}$$

because the derivative of at (with respect to t) is a. Note that $v(0) = 0$; i.e., the car starts from rest. At $t = 3\,$s, the velocity is the final velocity,

$$v_{\mathrm{f}} = v(3\,\mathrm{s}) = 5\,\frac{\mathrm{m}}{\mathrm{s}^2} \times 3\,\mathrm{s} = 15\,\frac{\mathrm{m}}{\mathrm{s}}. \tag{14}$$

(b) The position as a function of time is $x(t)$, and

$$\frac{dx}{dt} = v(t). \tag{15}$$

Equation (15) is called a *differential equation* for $x(t)$. We know the derivative; what is the function? The general methods for solving differential equations involve integration.[3] But for this simple case we can figure out the answer by guess-work.

Before we try to do the calculation, let's make sure we understand the problem. The car moves with velocity $v(t) = at$ (constant acceleration) for 3 seconds. How far does the car move during that time? Thereafter it moves with constant velocity $v_{\mathrm{f}} = 15\,$m/s. How far does it move from $t = 3\,$s to $t = 10\,$s? The combined distance is the distance traveled in $10\,$s.

For $t < 3\,$s the velocity is $v(t) = at$. This must be the derivative of $x(t)$. What function $x(t)$ has derivative at? The derivative is also a power of t, so $x(t)$ is also a power; recall that the derivative of t^p is pt^{p-1}. If we set $p = 2$ then the derivative is $\propto t$, as required. Multiplying by a constant factor does not change the power of t. Evidently $x(t)$ should be Ct^2 for some constant C. Then $dx/dt = 2Ct$. This is supposed to be at; thus, $C = a/2$. So the distance at t, for $t < 3\,$s, is $\frac{1}{2}at^2$. We could also add a constant x_0 because the derivative of a constant is 0. So, more generally, $x(t) = x_0 + \frac{1}{2}at^2$. But the stop sign is at $x = 0$, so $x(0) = 0$; that initial condition requires $x_0 = 0$. Hence the position as a function of t, for $t < 3\,$s, is

$$x(t) = \tfrac{1}{2}at^2. \qquad \text{(for } t \le 3\,\mathrm{s)} \tag{16}$$

For $t > 3\,$s the velocity is constant v_{f}. What function $x(t)$ has derivative v_{f}? By similar reasoning,

$$x(t) = c + v_{\mathrm{f}}t \qquad \text{(for } t \ge 3\,\mathrm{s)} \tag{17}$$

---

[3]We'll begin to study the integral and integration in Chapter 12.

general equations for constant acceleration
position $\quad x(t) = x_0 + v_0 t + \frac{1}{2}at^2$
velocity $\quad v(t) = dx/dt = v_0 t + at$
acceleration $\quad a(t) = dv/dt = a$, constant

Table 1: Formulae for constant acceleration. (x_0 = initial position, v_0 = initial velocity, and a = acceleration.)

where c is a constant. Both (16) and (17) must be correct. So, if we set $t = t_1 \equiv 3\,\mathrm{sec}$ in the two equations, we must have

$$x(t_1) = \tfrac{1}{2}at_1^2 = c + v_\mathrm{f}t_1. \tag{18}$$

(Remember, $t_1 = 3\,\mathrm{s}$ is the time when the car ceases to accelerate.) The only unknown in (18) is c, and solving the equation gives

$$
\begin{aligned}
c &= \tfrac{1}{2}at_1^2 - v_\mathrm{f}t_1 \tag{19}\\
&= \frac{1}{2}\left(5\,\mathrm{m/s^2}\right)(3\,\mathrm{s})^2 - (15\,\mathrm{m/s})(3\,\mathrm{s}) = -22.5\,\mathrm{m}. \tag{20}
\end{aligned}
$$

The position of the car at $t = 10\,\mathrm{s}$ is now obtained from (17),

$$
\begin{aligned}
x(10\,\mathrm{s}) &= c + v_\mathrm{f} \times 10\,\mathrm{s}\\
&= -22.5\,\mathrm{m} + 15\frac{\mathrm{m}}{\mathrm{s}} \times 10\,\mathrm{s} = 127.5\,\mathrm{m}. \tag{21}
\end{aligned}
$$

After $10\,\mathrm{s}$ the car has moved $127.5\,\mathrm{m}$.

Generalization. Some useful general formulae for constant acceleration are recorded in Table 1. In the table, v_0 is a constant equal to the velocity at $t = 0$. Also, x_0 is a constant equal to the position at $t = 0$. As an exercise, please verify that $a = dv/dt$ and $v = dx/dt$. Remember that x_0 and v_0 are constants, so their derivatives are 0. The velocity and position as functions of t for constant acceleration are illustrated in Fig. 3.

Example 4. A stone is dropped from a diving platform $10\,\mathrm{m}$ high. When does it hit the water? How fast is it moving then?

Solution. We'll denote the height above the water surface by $y(t)$. The initial height is $y_0 = 10\,\mathrm{m}$. The initial velocity is v_0; because the stone is dropped, not thrown, its initial velocity v_0 is 0. The acceleration of an

object in Earth's gravity, neglecting the effects of air resistance,[4] is $a = -g$ where $g = 9.8 \, \text{m/s}^2$. The acceleration a is negative because the direction of acceleration is downward; i.e., the stone accelerates toward smaller y. Using Table 1, the equation for position y as a function of t is

$$y(t) = y_0 - \tfrac{1}{2}gt^2. \tag{22}$$

The variable y is the height above the water, so the surface is at $y = 0$. The time t_f when the stone hits the water is obtained by solving $y(t_\text{f}) = 0$,

$$y_0 - \tfrac{1}{2}gt_\text{f}^2 = 0. \tag{23}$$

The time is

$$t_\text{f} = \sqrt{\frac{2y_0}{g}} = \sqrt{\frac{2 \times 10 \, \text{m}}{9.8 \, \text{m/s}^2}} = 1.43 \, \text{s}. \tag{24}$$

Note how the final unit came out to be seconds, which is correct. The time to fall to the water surface is $1.43 \, \text{s}$.

The equation for velocity is

$$v(t) = \frac{dy}{dt} = -gt. \tag{25}$$

This is consistent with the second row in Table 1, because $v_0 = 0$ and $a = -g$. The velocity when the stone hits the water is

$$v_\text{f} = v(t_\text{f}) = -9.8 \, \frac{\text{m}}{\text{s}^2} \times 1.43 \, \text{s} = -14.0 \, \frac{\text{m}}{\text{s}}. \tag{26}$$

The velocity is negative because the stone is moving downward. The final *speed* is the absolute value of the velocity, $14.0 \, \text{m/s}$.

2.3   Newton's second law

Newton's second law of motion states that the acceleration $\mathbf{a}$ of an object is proportional to the *net force* $\mathbf{F}$ acting on the object,

$$\mathbf{a} = \frac{\mathbf{F}}{m}, \tag{27}$$

or $\mathbf{F} = m\mathbf{a}$. The constant of proportionality m is the *mass* of the object. Equation (27) may be taken as the *definition* of the quantity m, the mass.

---

[4]Air resistance is a frictional force called "drag," which depends on the size, shape, surface roughness, and speed of the moving object. The effect on a stone falling $10 \, \text{m}$ is small.

>**Vectors.** For two- or three-dimensional motion, the position, velocity, and accleration are all vectors—mathematical quantities with both magnitude and direction. We will denote vectors by **boldface** symbols, e.g., **x** for position, **v** for velocity, and **a** for acceleration. In hand-written equations, vector quantities are usually indicated by drawing an arrow ($\rightarrow$) over the symbol.

Acceleration is a kinematic quantity—determined by the motion. Equation (27) relates acceleration and force. *But some other theory must determine the force.* There are only a few basic forces in nature: gravitational, electric and magnetic, and nuclear. All observed forces (e.g., contact, friction, a spring, atomic forces, etc.) are produced in some way by those basic forces. Whatever force is acting in a system, (27) states how that force influences the motion of a mass m (in classical mechanics!).

The mass in (27) is called the *inertial mass*, because it would be determined by measuring the acceleration produced by a given force. For example, if an object is pulled by a spring force of $50\,\mathrm{N}$, and the resulting acceleration is measured to be $5\,\mathrm{m/s^2}$, then the mass is equal to $10\,\mathrm{kg}$.

The gravitational force is exactly (i.e., as precisely as we can measure it!) proportional to the inertial mass. Therefore the acceleration due to gravity is independent of the mass of the accelerating object. Near the surface of the Earth, all falling objects have the same acceleration due to gravity, $g = 9.8\,\mathrm{m/s^2}$, neglecting the force of air resistance.[5] It took the great genius of Galileo to see that the small differences between falling objects are not produced by gravity but by air resistance. The force of gravity is proportional to the mass so accelerations by gravity are independent of the mass.

The equation $\mathbf{F} = m\mathbf{a}$ tells an engineer how an object will respond to a specified force. Because the acceleration $\mathbf{a}$ is a derivative,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{x}}{dt^2},$$

Newton's second law is a *differential equation*.

---

[5]Take a sheet of paper and drop it. It falls slowly and irregularly, not moving straight down but fluttering this way and that, because of aerodynamic forces. But wad the same piece of paper up into a small ball and drop it. Then it falls with the same acceleration as a more massive stone.

3 PROJECTILE MOTION

A projectile is an object M moving in Earth's gravity with no internal propulsion, and no external forces except gravity. (A *real* projectile is also subject to aerodynamic forces such as drag and lift. We will neglect these forces, a fairly good approximation if M moves slowly.)

The motion of a projectile must be described with *two* coordinates: horizontal (x) and vertical (y). Figure 4 shows the motion of the projectile in the xy coordinate system. The curve is the *trajectory* of M.

Suppose M is released at $(x, y) = (x_0, y_0)$ at time $t = 0$. Figure 4 also indicates the initial velocity vector $\mathbf{v}_0$ which is tangent to the trajectory at (x_0, y_0). Let θ be the angle of elevation of the initial velocity; then the x and y components of the initial velocity vector are

$$
\begin{aligned}
v_{0x} &= v_0 \cos\theta, & (28) \\
v_{0y} &= v_0 \sin\theta. & (29)
\end{aligned}
$$

Horizontal component of motion.   The equations for the horizontal motion are

$$
\begin{aligned}
x(t) &= x_0 + v_{0x}t, & (30) \\
v_x(t) &= v_{0x}. & (31)
\end{aligned}
$$

These are the equations for constant velocity, $v_x = v_{0x}$. There is no horizontal acceleration (neglecting air resistance) because the gravitational force is vertical.

Vertical component of motion.   The equations for the vertical motion are

$$
\begin{aligned}
y(t) &= y_0 + v_{0y}t - \tfrac{1}{2}gt^2, & (32) \\
v_y(t) &= v_{0y} - gt. & (33)
\end{aligned}
$$

These are the equations for constant acceleration, $a_y = -g$. (As usual, $g = 9.8\,\text{m/s}^2$.) The vertical force is $F_y = -mg$, negative indicating downward, where m is the mass of the projectile. The acceleration is $a_y = F_y/m$ by Newton's second law.[6] Thus $a_y = -g$. The acceleration due to gravity does not depend on the mass of the projectile because the force is proportional to the mass.

Example 5. Verify that the derivative of the position vector is the velocity vector, and the derivative of the velocity vector is the acceleration vector, for

---

[6]Newton's second law is $\mathbf{F} = m\mathbf{a}$.

the projectile.

Solution. Position, velocity, and acceleration are all *vectors*. The position vector is

$$\mathbf{x}(t) = x(t)\widehat{\mathbf{i}} + y(t)\widehat{\mathbf{j}}. \tag{34}$$

Here $\widehat{\mathbf{i}}$ denotes the horizontal unit vector, and $\widehat{\mathbf{j}}$ denotes the vertical unit vector (cf. Fig. 5). For the purposes of describing the motion, these unit vectors are *constants*, independent of t. The time dependence of the position vector $\mathbf{x}(t)$ is contained in the coordinates, $x(t)$ and $y(t)$.

The derivative of $\mathbf{x}(t)$ is

$$\begin{aligned}
\frac{d\mathbf{x}}{dt} &= \frac{dx}{dt}\widehat{\mathbf{i}} + \frac{dy}{dt}\widehat{\mathbf{j}} \\
&= v_{0x}\widehat{\mathbf{i}} + (v_{0y} - gt)\widehat{\mathbf{j}} \\
&= v_x\widehat{\mathbf{i}} + v_y\widehat{\mathbf{j}} = \mathbf{v}.
\end{aligned} \tag{35}$$

As required, $d\mathbf{x}/dt$ is $\mathbf{v}$. The derivative of $\mathbf{v}(t)$ is

$$\begin{aligned}
\frac{d\mathbf{v}}{dt} &= \frac{dv_x}{dt}\widehat{\mathbf{i}} + \frac{dv_y}{dt}\widehat{\mathbf{j}} \\
&= 0 + (-g)\widehat{\mathbf{j}} = -g\widehat{\mathbf{j}}.
\end{aligned} \tag{36}$$

The acceleration vector has magnitude g and direction $-\widehat{\mathbf{j}}$, i.e., downward; so $\mathbf{a} = -g\widehat{\mathbf{j}}$. We see that $d\mathbf{v}/dt = \mathbf{a}$, as required.

3.0.1 Summary

To describe projectile motion (or 3D motion in general) we must use vectors. However, for the ideal projectile (without air resistance) the two components— horizontal and vertical—are independent. The horizontal component of the motion has constant velocity v_{0x}, leading to Eqs. (30) and (31). The vertical component of the motion has constant acceleration $a_y = -g$, leading to Eqs. (32) and (33).

To depict the motion, we could plot $x(t)$ and $y(t)$ versus t separately, or make a parametric plot of y versus x with t as independent parameter.[7] The parametric plot yields a parabola. Galileo was the first person to understand the trajectory of an ideal projectile (with negligible air resistance): The trajectory is a parabola.

---

[7]Exercise 11.

4   CIRCULAR MOTION

Consider an object M moving on a circle of radius R, as illustrated in Fig. 6.
We could describe the motion by Cartesian coordinates, $x(t)$ and $y(t)$, but it
is simpler to use the angular position $\theta(t)$ because the radius R is constant.
The angle θ is defined in Fig. 6. It is the angle between the radial vector and
the x axis. The value of θ is sufficient to locate M. From Fig. 6 we see that
the Cartesian coordinates are

$$
\begin{align}
x(t) &= R\cos\theta(t), \tag{37}\\
y(t) &= R\sin\theta(t). \tag{38}
\end{align}
$$

If $\theta(t)$ is known, then $x(t)$ and $y(t)$ can be calculated from these equations.

In calculus we always use the *radian measure* for an angle θ. The radian
measure is defined as follows. Consider a circular arc with arclength s on a
circle of radius R. The angle subtended by the arc, in radians, is

$$\theta = \frac{s}{R}. \qquad \text{(radian measure)} \tag{39}$$

4.1   Angular velocity and the velocity vector

The angular velocity $\omega(t)$ is defined by

$$\omega(t) = \frac{d\theta}{dt}. \qquad \text{(angular velocity)} \tag{40}$$

This function is the *instantaneous* angular velocity at time t. For example,
if M moves with constant speed, traveling around the circle in time T, then
the angular velocity is constant and given by

$$\omega = \frac{2\pi}{T}. \qquad \text{(constant angular velocity)} \tag{41}$$

To derive this result, consider the motion during a time interval Δt. The
arclength Δs traveled along the circle during Δt is $R\Delta\theta$ where $\Delta\theta$ is the
change of θ during Δt, in radians. The angular velocity is then

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{\Delta s/R}{\Delta t}. \tag{42}$$

Because the speed is constant, $\Delta\theta/\Delta t$ is constant and independent of the
time interval Δt. Let Δt be one period of revolution T. The arclength for a
full revolution is the circumference $2\pi R$. Thus

$$\omega = \frac{2\pi R/R}{T} = \frac{2\pi}{T}. \tag{43}$$

The instantaneous *speed* of the object is the rate of increase of distance with time,

$$
\begin{aligned}
v(t) &= \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{R\Delta\theta}{\Delta t} \\
&= R\frac{d\theta}{dt} = R\omega(t).
\end{aligned}
\tag{44}
$$

But what is the instantaneous *velocity*? Velocity is a vector $\mathbf{v}$, with both direction and magnitude. The magnitude of $\mathbf{v}$ is the speed, $v = R\omega$. The direction is tangent to the circle, which is the same as the unit vector $\widehat{\boldsymbol{\theta}}$. (See Fig. 6.) Thus the velocity vector is

$$
\mathbf{v} = R\omega\widehat{\boldsymbol{\theta}},
\tag{45}
$$

which points in the direction of $\widehat{\boldsymbol{\theta}}$ and has magnitude $R\omega$. In general, $\mathbf{v}$, ω, θ and $\widehat{\boldsymbol{\theta}}$ are all functions of time t as the particle moves around the circle. But of course for circular motion, R is constant. We summarize our analysis as a theorem:

Theorem 1. The velocity vector in circular motion is

$$
\mathbf{v}(t) = R\,\omega(t)\,\widehat{\boldsymbol{\theta}}(t).
\tag{46}
$$

4.2   Acceleration in circular motion

Now, what is the *acceleration* of M as it moves on the circle? The acceleration $\mathbf{a}$ is a vector, so we must determine both its magnitude and direction. Unlike the velocity $\mathbf{v}$, which must be tangent to the circle, the acceleration has both tangential and radial components.

Recall that we have defined acceleration as the derivative of velocity in the case of one-dimensional motion. The same definition applies to the vector quantities for two- or three-dimensional motion. Using the definition of the derivative,

$$
\mathbf{a}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t + \Delta t) - \mathbf{v}(t)}{\Delta t} = \frac{d\mathbf{v}}{dt}.
\tag{47}
$$

The next theorem relates $\mathbf{a}$ for circular motion to the parameters of the motion.

Theorem 2. The acceleration vector in circular motion is

$$
\mathbf{a} = R\frac{d\omega}{dt}\,\widehat{\boldsymbol{\theta}} - R\omega^2\,\widehat{\mathbf{r}}.
\tag{48}
$$

Proof: We must calculate the derivative in (48) using (46) for $\mathbf{v}$. At time t, the acceleration is

$$
\begin{aligned}
\mathbf{a}(t) & = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(R\omega\widehat{\boldsymbol{\theta}}\right) \\
& = R\left(\frac{d\omega}{dt}\widehat{\boldsymbol{\theta}} + \omega\frac{d\widehat{\boldsymbol{\theta}}}{dt}\right).
\end{aligned}
\tag{49}
$$

Note that (49) follows from the Leibniz rule for the derivative of the product $\omega(t)\widehat{\boldsymbol{\theta}}(t)$. Now,

$$
\frac{d\widehat{\boldsymbol{\theta}}}{dt} = \lim_{\Delta t \to 0}\frac{\Delta\widehat{\boldsymbol{\theta}}}{\Delta t}.
\tag{50}
$$

Figure 7 demonstrates that $\Delta\widehat{\boldsymbol{\theta}} \approx -\widehat{\mathbf{r}}\Delta\theta$ for small $\Delta\theta$. (The relation of differentials is $d\widehat{\boldsymbol{\theta}} = -\widehat{\mathbf{r}}\,d\theta$.) The direction of $\Delta\widehat{\boldsymbol{\theta}}$ is *radially inward*. This little result has interesting consequences, as we'll see! The derivative is then

$$
\frac{d\widehat{\boldsymbol{\theta}}}{dt} = \lim_{\Delta t \to 0}\frac{-\widehat{\mathbf{r}}\Delta\theta}{\Delta t} = -\widehat{\mathbf{r}}\frac{d\theta}{dt} = -\widehat{\mathbf{r}}\omega.
\tag{51}
$$

Substituting this result into (49) we find

$$
\mathbf{a}(t) = R\frac{d\omega}{dt}\,\widehat{\boldsymbol{\theta}} - R\omega^2\,\widehat{\mathbf{r}},
\tag{52}
$$

which proves the theorem.

The radial component of the acceleration vector is $a_r = -R\omega^2$. This component of $\mathbf{a}$ is called the *centripetal acceleration*. The word "centripetal" means *directed toward the center*. We may write a_r in another form. By Theorem 9-1, $\omega = v/R$; therefore

$$
a_r = -\frac{v^2}{R}.
\tag{53}
$$

If the speed of the object is constant, then $d\omega/dt = 0$ and the acceleration $\mathbf{a}$ is purely centripetal. In *uniform circular motion*, the acceleration vector is always directed toward the center of the circle with magnitude v^2/R.

Imagine a ball attached to a string of length R, moving around a circle at constant speed with the end of the string fixed. The trajectory must be a circle because the string length (the distance from the fixed point) is constant. *The ball constantly accelerates toward the center of the circle ($a_r = -v^2/R$) but it never gets any closer to the center ($r(t) = R$, constant)!* This example illustrates the fact that the velocity and acceleration vectors may point in different directions. In uniform circular motion, the velocity is tangent to the

circle but the acceleration is centripetal, i.e., orthogonal to the velocity.

Example 6. Suppose a race car travels on a circular track of radius $R = 50\,\mathrm{m}$. (This is quite small!) At what speed is the centripetal acceleration equal to $1\,g$?

Solution. Using the formula $a = v^2/R$, and setting $a = g$, the speed is

$$v = \sqrt{gR} = \sqrt{9.8\,\mathrm{m/s^2} \times 50\,\mathrm{m}} = 22.1\,\mathrm{m/s}. \tag{54}$$

Converting to miles per hour, the speed is about $48\,\mathrm{mi/hr}$. A pendulum suspended from the ceiling of the car would hang at an angle of 45 degrees to the vertical (in equilibrium), because the horizontal and vertical components of force exerted by the string on the bob would be equal, both equal to mg. The pendulum would hang *outward* from the center of the circle, as shown in Fig. 8. Then the string exerts a force on the bob with an *inward* horizontal component, which is the centripetal force on the bob.

$$\star$$

The equation $a_r = -v^2/r$ for the centripetal acceleration in circular motion was first published by Christiaan Huygens in 1673 in a book entitled *Horologium Oscillatorium*. Huygens, a contemporary of Isaac Newton, was one of the great figures of the scientific revolution. He invented the earliest practical pendulum clocks (the main subject of the book mentioned). He constructed excellent telescopes, and discovered that the planet Saturn is encircled by rings. In his scientific work, Huygens was guided by great skill in mathematical analysis. Like Galileo and Newton, Huygens used mathematics to describe nature accurately.

5   KEPLER'S LAWS OF PLANETARY MOTION

Kepler's first law is that the planets travel on ellipses with the sun at one focal point. Newton deduced from this empirical observation that the gravitational force on the planet must be proportional to $1/r^2$ where r is the distance from the sun.

Figure 9 shows a possible planetary orbit. The ellipse is characterized by two parameters: a = semimajor axis and e = eccentricity.

5.1   Kepler's third law

Kepler's third law relates the period T and the semimajor axis a of the ellipse. To the accuracy of the data available in his time, Kepler found that T^2 is proportional to a^3. The next example derives this result from Newtonian mechanics, for the special case of a circular orbit. A circle is an ellipse with eccentricity equal to zero; then the semimajor axis is the radius.

Example 7. Show that $T^2 \propto r^3$ for a planet in a circular orbit of radius r around the sun.[8]

Solution. In analyzing the problem, we will neglect the motion of the sun. More precisely, the sun and planet both revolve around their center of mass. But because the sun is much more massive than the planet, the center of mass is approximately at the position of the sun, so that the sun may be considered to be at rest. Then the planet moves on a circle around the sun. Let m denote the mass of the planet, and M the mass of the sun.

For a circular orbit the angular speed of the planet is constant, $d\omega/dt = 0$. Therefore the acceleration is $\mathbf{a} = -r\omega^2\widehat{\mathbf{r}}$. In terms of the speed $v = r\omega$,

$$\mathbf{a} = -\frac{v^2}{r}\widehat{\mathbf{r}}. \tag{55}$$

The direction is $-\widehat{\mathbf{r}}$, i.e., centripetal, toward the sun. The gravitational force exerted by the sun on the planet is

$$\mathbf{F} = -\frac{GMm}{r^2}\widehat{\mathbf{r}}, \tag{56}$$

which is also centripetal. Equation (56) is Newton's theory of Universal Gravitation, in which the force is proportional to $1/r^2$.

Newton's second law of motion states that $\mathbf{F} = m\mathbf{a}$. Therefore,

$$\frac{mv^2}{r} = \frac{GMm}{r^2}. \tag{57}$$

[8]We consider an ideal case in which the *other* planets have a negligible effect on the planet being considered. This is a good approximation for the solar system, but not exact.

The distance traveled in time T (one period of revolution) is $2\pi r$ (the circumference of the orbit), so the speed is $v = 2\pi r/T$. Substituting this expression for v into (57) gives

$$\left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r}.\tag{58}$$

Or, rearranging the equation,

$$T^2 = \frac{4\pi^2 r^3}{GM};\tag{59}$$

we see that T^2 is proportional to r^3, as claimed.

In obtaining (59) we neglected the small motion of the sun around the center-of-mass point. This is a very good approximation for the solar system. In this approximation, T^2/r^3 is constant, i.e., *it has the same value for all nine planets.*

5.2   Kepler's second law

Kepler's second law is that the radial vector sweeps out equal areas in equal times. This law is illustrated in Fig. 10. In Newtonian mechanics it is a consequence of conservation of angular momentum. The next two examples show how Kepler's second law follows from Newton's theory.

***Example 8.* Conservation of angular momentum**
The angular momentum L of an object of mass m moving in the xy plane is defined by

$$L = m\left(xv_y - yv_x\right).\tag{60}$$

Show that L is constant if the force on the object is central.

Solution.   To show that a function is constant, we must show that its derivative is 0.[9] In (60), the coordinates x and y, and velocity components v_x and v_y, are all functions of time t. But the particular combination in L is constant, as we now show. The derivative of L is

$$
\begin{aligned}
\frac{dL}{dt} &= m\left[\frac{dx}{dt}v_y + x\frac{dv_x}{dt} - \frac{dy}{dt}v_x - y\frac{dv_x}{dt}\right]\\
&= mv_xv_y + xF_y - mv_yv_x - yF_x\\
&= xF_y - yF_x.
\end{aligned}\tag{61}
$$

In the first step we have used the fact that $d\mathbf{x}/dt = \mathbf{v}$, and $d\mathbf{v}/dt = \mathbf{a}$; also,

---

[9]The derivative of any constant is 0.

by Newton's second law, the acceleration $\mathbf{a}$ is equal to $\mathbf{F}/m$. The final line (61) is called the torque on the object.

For any central force the torque is 0. What is meant by the term "central force" is that the force is in the direction of $\pm\widehat{\mathbf{r}}$, i.e., along the line to the origin. (The sign—attractive or repulsive— is unimportant for the proof of conservation of angular momentum.) Figure 11 shows a central force $\mathbf{F}$ toward the origin. The components of $\mathbf{F}$ are

$$F_x = -F\cos\theta \quad \text{and} \quad F_y = -F\sin\theta \tag{62}$$

where F is the strength of the force and the minus signs mean that $\mathbf{F}$ is toward 0. Thus the torque on the object is

$$
\begin{aligned}
\text{torque} \quad &= \quad xF_y - yF_x \\
&= \quad -r\cos\theta\,F\sin\theta + r\sin\theta\,F\cos\theta = 0.
\end{aligned}
\tag{63}
$$

Since the torque is 0, equation (61) implies that $dL/dt = 0$. Since the derivative is 0, the angular momentum L is constant, as claimed.

Example 9. Kepler's law of equal areas

Show that the radial vector from the sun to a planet sweeps out equal areas in equal times.

Solution. Figure 10(a) shows the elliptical orbit. The shaded area ΔA is the area swept out by the radial vector between times t and $t + \Delta t$. The shaded area may be approximated by a triangle, with base r and height $r\Delta\theta$, where $\Delta\theta$ is the change of the angular position between t and $t + \Delta t$. Approximating the area as a triangle is a good approximation for small Δt. Now consider the limit $\Delta t \to 0$; i.e., Δt and ΔA become the differentials dt and dA. The area of the triangle becomes

$$dA = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times r \times rd\theta = \tfrac{1}{2}r^2 d\theta. \tag{64}$$

Thus, in the limit $\Delta t \to 0$, where we replace Δt by dt,

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \tfrac{1}{2}r^2\omega. \tag{65}$$

We'll use this result presently.

But now we must express the angular momentum in polar coordinates. The position vector of M is $\mathbf{x} = r\widehat{\mathbf{r}}$, and its x and y components are

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta. \tag{66}$$

The velocity vector is

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \frac{dr}{dt}\widehat{\mathbf{r}} + r\frac{d\theta}{dt}\widehat{\boldsymbol{\theta}}, \tag{67}$$

because $d\widehat{\mathbf{r}} = \widehat{\boldsymbol{\theta}}\, d\theta$.[10] The x and y components of velocity are

$$v_x = \frac{dr}{dt}\cos\theta - r\frac{d\theta}{dt}\sin\theta, \tag{68}$$

$$v_y = \frac{dr}{dt}\sin\theta + r\frac{d\theta}{dt}\cos\theta. \tag{69}$$

L is defined in (60); substituting the polar expressions for x, y, v_x and v_y we find

$$\begin{aligned}
L &= m\left(xv_y - yv_x\right) \\
&= m\left[r\frac{dr}{dt}\cos\theta\sin\theta + r^2\frac{d\theta}{dt}\cos^2\theta\right] \\
&\quad -m\left[r\frac{dr}{dt}\sin\theta\cos\theta - r^2\frac{d\theta}{dt}\sin^2\theta\right] \\
&= mr^2\frac{d\theta}{dt}\left(\cos^2\theta + \sin^2\theta\right) = mr^2\frac{d\theta}{dt}.
\end{aligned} \tag{70}$$

The result is

$$L = mr^2\omega. \tag{71}$$

Comparing this to (65) we see that dA/dt is $L/2m$. But L is a constant of the motion by conservation of angular momentum. Thus dA/dt is constant. In words, *the rate of change of the area is constant*, i.e., independent of position on the orbit. Hence Kepler's second law is explained: The area increases at a constant rate, so equal areas are swept out in equal times.

5.3   The inverse square law

Kepler's first law is that the planets travel on ellipses with the sun at one focal point. We will prove that this observation implies that the force on the planet must be an inverse square law, i.e., proportional to $1/r^2$ where r is the distance from the sun. The calculations depend on all that we have learned about derivatives and differentiation.

The equation for an elliptical orbit in polar coordinates (r, θ) is

$$r(\theta) = \frac{a(1 - e^2)}{1 + e\cos\theta} \tag{72}$$

where a = semimajor axis and e = eccentricity. Figures 9 and 10 show graphs

---

[10]Exercise 20.

of the ellipse. What force is implied by the orbit equation (72)? The radial acceleration is[11]

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 . \tag{73}$$

The first term involves the change of radius; the second term is the centripetal acceleration $-r\omega^2$. Now, a_r must equal F_r/m by Newton's second law. To determine the radial force F_r we must express a_r as a function of r. We know that angular momentum is constant; by (71),

$$mr^2\frac{d\theta}{dt} = L, \qquad \text{so} \qquad \frac{d\theta}{dt} = \frac{L}{mr^2}. \tag{74}$$

Now starting from (72), and applying the chain rule,[12]

$$\begin{aligned}
\frac{dr}{dt} &= \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{-a(1-e^2)}{(1+e\cos\theta)^2}(-e\sin\theta)\frac{d\theta}{dt} \\
&= \frac{a(1-e^2)e\sin\theta}{(1+e\cos\theta)^2}\frac{L(1+e\cos\theta)^2}{ma^2(1-e^2)^2} = \frac{Le\sin\theta}{ma(1-e^2)};
\end{aligned} \tag{75}$$

and, taking another derivative,

$$\frac{d^2r}{dt^2} = \frac{Le\cos\theta}{ma(1-e^2)}\frac{d\theta}{dt} = \frac{Le\cos\theta}{ma(1-e^2)}\frac{L}{mr^2}. \tag{76}$$

Combining these results in (73), the radial component of the acceleration is

$$\begin{aligned}
a_r &= \frac{L^2 e\cos\theta}{m^2 a(1-e^2)r^2} - r\left(\frac{L}{mr^2}\right)^2 \\
&= \frac{L^2}{m^2 r^2}\left\{\frac{e\cos\theta}{a(1-e^2)} - \frac{1+e\cos\theta}{a(1-e^2)}\right\} = \frac{-L^2}{m^2 a(1-e^2)r^2}.
\end{aligned} \tag{77}$$

By Newton's second law, then, the radial force must be

$$F_r = ma_r = -\frac{k}{r^2} \qquad \text{where} \qquad k = \frac{L^2}{ma(1-e^2)}. \tag{78}$$

Our result is that the force on the planet must be an attactive inverse-square-law, $F_r = -k/r^2$. The orbit parameters are related to the force parameter k by

$$L^2 = ma(1-e^2)k. \tag{79}$$

---

[11] See Exercise 21.

[12] The calculations of (75) and (76) require these results from Chapter 11: the derivative (with respect to θ) of $\cos\theta$ is $-\sin\theta$, and the derivative of $\sin\theta$ is $\cos\theta$.

5.3.1   Newton's Theory of Universal Gravitation

From the fact that planetary orbits are elliptical, Newton deduced that $F_r = -k/r^2$. Also, k must be proportional to the planet's mass m because $T^2 \propto a^3$, independent of the mass (cf. Section 9.5.1). But then k must also be proportional to the solar mass, because for every action there is an equal but opposite reaction. Therefore the force vector must be

$$\mathbf{F} = F_r \widehat{\mathbf{r}} = -\frac{GMm}{r^2} \widehat{\mathbf{r}} \tag{80}$$

where G is a constant. Newton's theory of universal gravitation is that any two masses in the universe, m and M, attract each other according to the force (80).

Newton's gravitational constant G cannot be determined by astronomical observations, because the solar mass M is not known independently. G must be measured in the laboratory. An accurate measurement of G is very difficult, and was not accomplished in the time of Newton. The first measurement of G was by Henry Cavendish in 1798. G is hard to measure because gravity is extremely weak,

$$G = 6.67 \times 10^{-11} \, \mathrm{m}^3 \, \mathrm{s}^{-2} \, \mathrm{kg}^{-1}. \tag{81}$$

Newton's theory of gravity is very accurate, but not exact. A more accurate theory of gravity—the theory of general relativity—was developed by Einstein. In relativity, planetary orbits are not perfect ellipses; the orbits *precess* very slowly. Indeed this precession is observed in precise measurements of planetary positions, and the measurements agree with the relativistic calculation.

$$\star \qquad \star \qquad \star$$

The examples in this chapter show how calculus is used to understand profound physical observations such as the motion of the planets. That is why calculus is so important in the education of scientists and engineers.
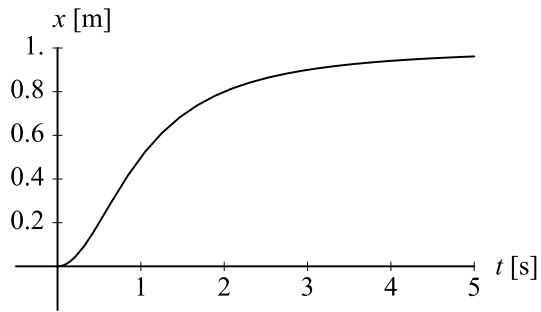
Figure 1: An example of position x as a function of time t for an object moving in one dimension. The object starts at rest at the origin at $t = 0$, begins moving to positive x, has positive acceleration for about 1 second, and then gradually slows to a stop at a distance of 1 m from the origin.
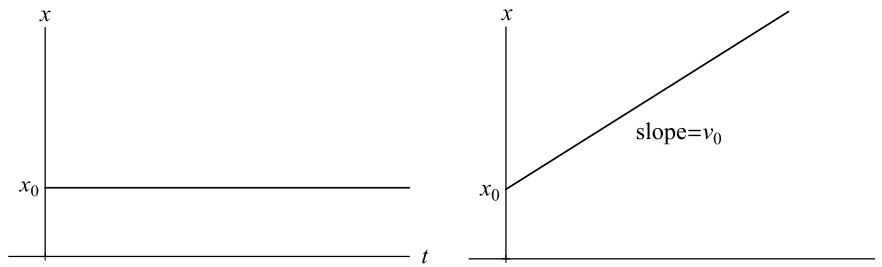

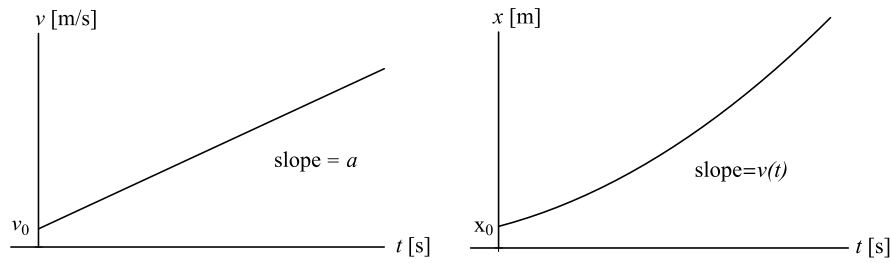Figure 2: Motion of an object: (a) zero velocity and (b) constant positive velocity.

Figure 3: **Constant acceleration.** The two graphs are (a) velocity $v(t)$ and (b) position $x(t)$ as functions of time, for an object with constant acceleration a.
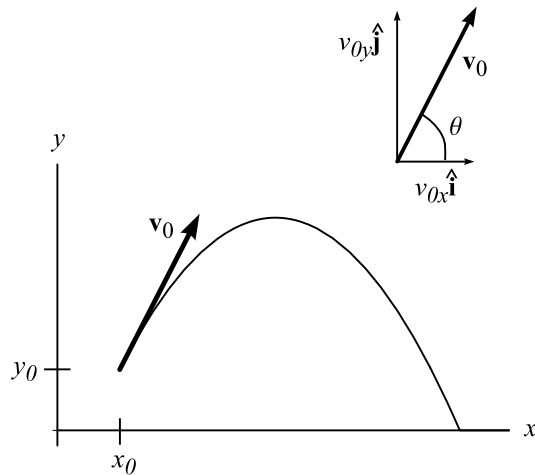
Figure 4: **Projectile motion.** Horizontal (x) and vertical (y) axes are set up to analyze the motion. The initial position is $(x, y) = (x_0, y_0)$. The initial velocity $\mathbf{v}_0$ is shown as a vector at (x_0, y_0). The curve is the trajectory of the projectile. The inset shows the initial velocity vector $\mathbf{v}_0$ separated into horizontal and vertical components, $v_{0x}\hat{\mathbf{i}}$ and $v_{0y}\hat{\mathbf{j}}$; θ is the angle of elevation of $\mathbf{v}_0$. ($\hat{\mathbf{i}} =$ unit horizontal vector, $\hat{\mathbf{j}} =$ unit vertical vector)
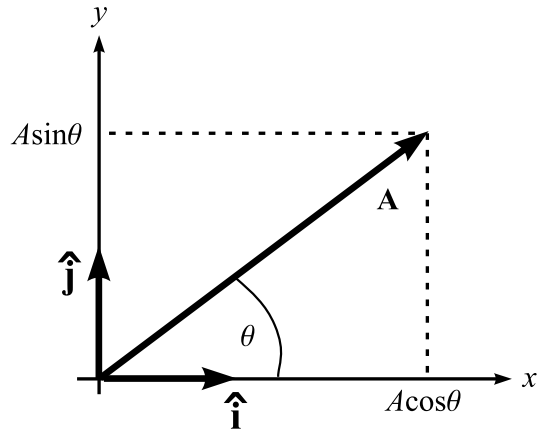
Figure 5: The unit vectors $\widehat{\mathbf{i}}$ and $\widehat{\mathbf{j}}$ point in the x and y directions, respectively. Any vector $\mathbf{A}$ may be written as $A_x\widehat{\mathbf{i}} + A_y\widehat{\mathbf{j}}$ where $A_x = A\cos\theta$ and $A_y = A\sin\theta$. The magnitude of the vector is $A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2}$.
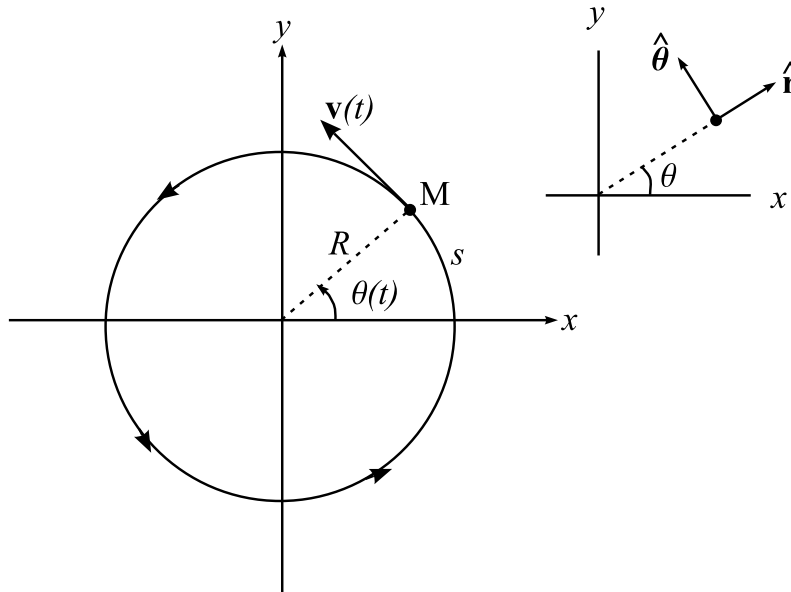
Figure 6: **Circular motion.** A mass M moves on a circle of radius R. The angle $\theta(t)$ is used to specify the position. In radians, $\theta = s/R$ where s is the arclength. The velocity vector $\mathbf{v}(t)$ is tangent to the circle. The inset shows the unit vectors $\widehat{\boldsymbol{\theta}}$ and $\widehat{\mathbf{r}}$, which point in the direction of increasing θ and r, respectively.
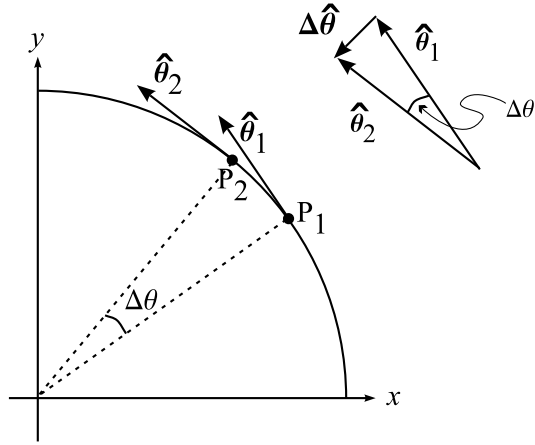
Figure 7: Proof that $d\widehat{\boldsymbol{\theta}} = -\widehat{\mathbf{r}}\,d\theta$. P_1 and P_2 are points on the circle with angle difference $\Delta\theta$. The inset shows that $\Delta\widehat{\boldsymbol{\theta}}\,(= \widehat{\boldsymbol{\theta}}_2 - \widehat{\boldsymbol{\theta}}_1)$ is centripetal (i.e., in the direction of $-\widehat{\mathbf{r}}$) and has magnitude $\Delta\theta$ in the limit of small $\Delta\theta$.
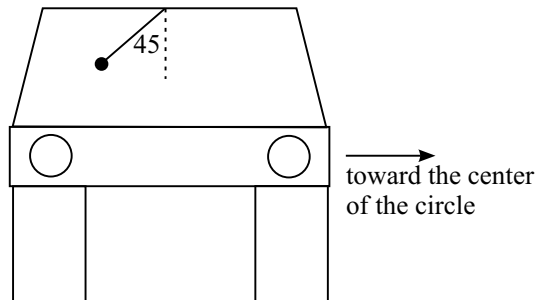

Figure 8: A race car on a circular track has centripetal acceleration v^2/R. If $v^2/R = g$, then the equilibrium of a pendulum suspended from the ceiling is at 45 degrees to the vertical. In the frame of reference of the track, the bob accelerates centripetally because it is pulled toward the center by the pendulum string. In the frame of reference of the car there is a centrifugal force—an apparent (but fictitious) force directed away from the center of the track.
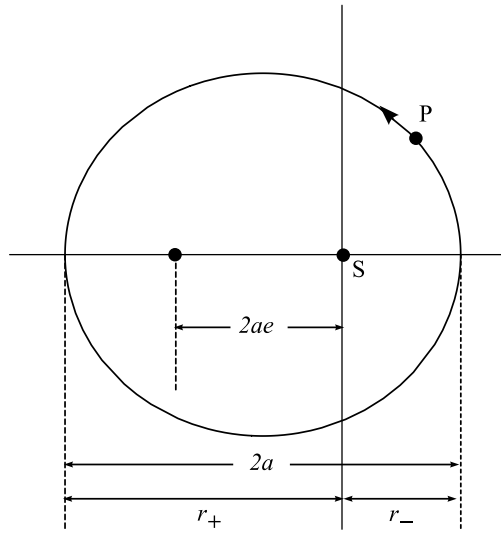
Figure 9: A possible planetary orbit. The sun S is at the origin, which is one focal point of the ellipse, and the planet P moves on the ellipse. The large diameter is $2a$, where a is called the semimajor axis. The distance between the foci is $2ae$ where e is called the eccentricity. The perihelion distance is $r_- = a(1-e)$ and the aphelion distance is $r_+ = a(1+e)$. A circle is an ellipse with $e = 0$.
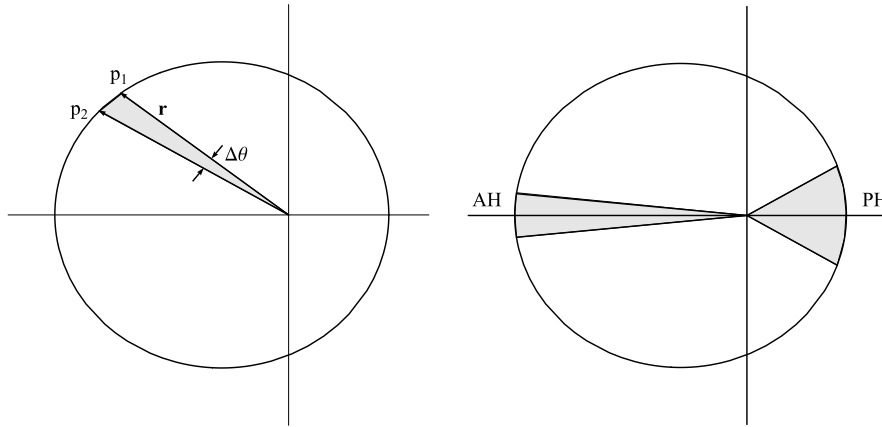

Figure 10: **Kepler's second law.** The radial vector sweeps out equal areas in equal times. (a) The radial vector sweeps out the shaded region as the planet moves from P_1 to P_2. (b) The planet moves faster near perihelion (PH) and slower near aphelion (AH).
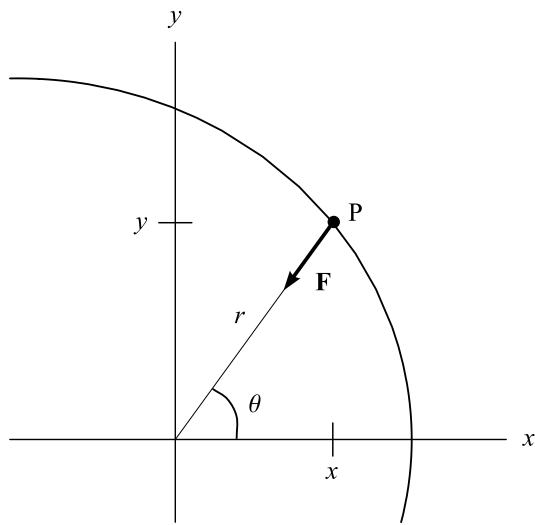
27

Figure 11: An attractive central force. The Cartesian coordinates at P are $x = r\cos\theta$ and $y = r\sin\theta$. The force components are $F_x = -F\cos\theta$ and $F_y = -F\sin\theta$ where F is the magnitude of the force vector. The torque, $xF_y - yF_x$, is 0.

<div align="center">EXERCISES</div>

Section 2: Position, velocity, and acceleration

1. Show that $\dot{x} = v$ and $\ddot{x} = a$ for the functions in Table 1.

2. Consider a car driving on a straight road at 60 mph (mi/hr).

(a) How far does it travel in 1 second?

(b) What is the speed in ft/s?

3. **Race car.** A race car accelerates on a drag strip from 0 to 60 mph in 6 seconds.

(a) What is the acceleration a? Express a in ft/sec^2.

(b) How far does the car travel in that 6 seconds? Express the answer in feet.

4. The graph in Fig. 12 shows the acceleration $a(t)$ of an object, as a function of time t. At $t = 0$ the velocity is 0. Make plots of velocity $v(t)$ and position $x(t)$. Put accurate scales on the axes. What is the final position?
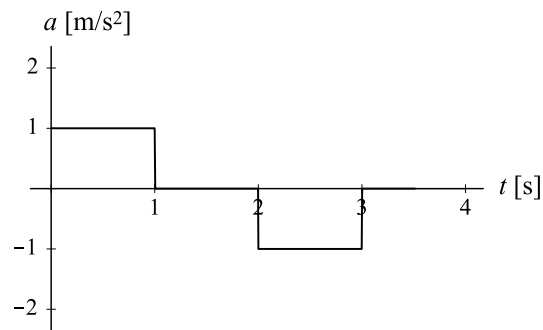

Figure 12: Exercise 4.

5. Consider the motion shown in the graph of position x as a function of time t, shown in Fig. 13. Sketch graphs of the velocity $v(t)$ and acceleration $a(t)$. In words, state what is happening between $t = 2$ and 3 s; between $t = 7$ and 7.75 s.

6. Suppose a mass $m = 3$ kg moves along the x axis according to the formula

$$x(t) = Ct^2(20 - t)^2$$

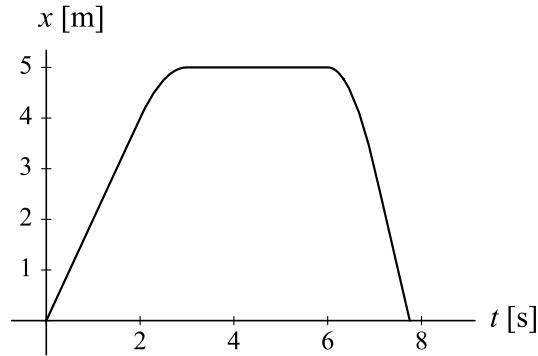Figure 13: Exercise 5.

for $0 \le t \le 20$. (The time t is measured in seconds.) The initial acceleration (at $t = 0$) is $4\,\mathrm{m/s^2}$.

(a) Determine C.

(b) Determine the velocity at $t = 5$, 10, 15, and 20 s.

(c) Find the force on the object, in newtons, at $t = 10\,\mathrm{s}$.

(d) Describe the motion in words.

7. Consider an object moving in one dimension as illustrated in Fig. 14.

(a) The acceleration between $t = 0$ and 3 s is constant. What is the acceleration?

(b) The acceleration between $t = 3$ and 4 s is constant. What is the acceleration?

(c) In words, what is happening for $t > 4\,\mathrm{s}$?

8. Conservation of energy is a unifying principle in science. For the dynamics of a particle moving in one dimension, the energy is

$$E = \tfrac{1}{2}mv^2 + U(x)$$

where $v = dx/dt$ and $U(x) =$ potential energy. Show that E is constant, i.e., $dE/dt = 0$. (Hints: The force and potential energy are related by $F(x) = -dU/dx$. Use the chain rule to calculate dU/dt, and remember Newton's second law.)

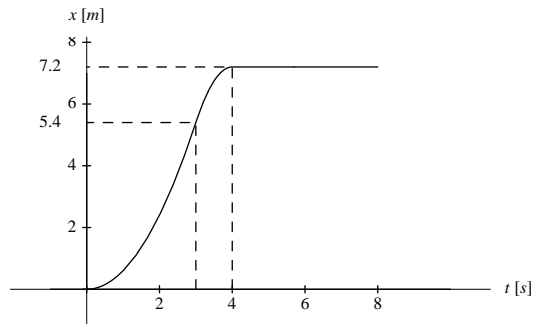9. Force $F(x)$ and potential energy $U(x)$, as functions of position x, are

Figure 14: Exercise 7

related by $F(x) = -dU/dx$. An object M is attached to one end of a spring, and the other end of the spring is attached to an immovable wall. The potential energy of the spring is $\frac{1}{2}kx^2$ where $x = $ displacement of M from the equilibrium. (x can be positive or negative.) Show that the force on M doubles as the displacement doubles. Show that the force is opposite in sign to the displacement. What does this imply about the direction of the force?

10. A stone is dropped from a tower. Let $y(t)$ be the height above the ground ($y = 0$ at ground level) as a function of time t. The gravitational potential energy is $U(y) = mgy$. Using the equations for constant acceleration $a_y = -g$, write a formula for total energy E as a function of time t. Is E constant?

Section 3: Projectiles

11. Consider projectile motion, neglecting air resistance. Sketch a graph of the horizontal coordinate $x(t)$ as a function of time t. Sketch a graph of the vertical coordinate $y(t)$ as a function of time t. Sketch a parametric plot of the vertical coordinate y versus the horizontal coordinate x, with t as the independent parameter. (Try using the parametric plot mode of a graphing calculator.) What curve is the graph of y versus x?

12. A ball is thrown horizontally at $50\,\mathrm{mi/hr}$ from a height of $5\,\mathrm{ft}$. Where will it hit the ground?

13. TBA

14. TBA

Section 4: Circular motion

15. Consider a go-cart moving on a circular track of radius $R = 40\,\text{m}$. Suppose it starts from rest and speeds up to $60\,\text{km/hr}$ in $20\,\text{seconds}$ (with constant acceleration). What is the acceleration vector at $t = 10\,\text{s}$? Give both the direction and magnitude.

16. Imagine a ball attached to a string of length R, moving along a circle at constant speed with the end of the string fixed. The ball constantly accelerates toward the center of the circle but it never gets any closer to the center. According to Newton's second law there *must be* a force in the direction of the acceleration. What is this force? Is it centripetal? If so, why?

17. TBA

Section 5: Motion of the planets

18. The Earth's orbit around the Sun is nearly circular, with radius $R_\oplus = 1.496 \times 10^{11}\,\text{m}$. From this, and the *laboratory* measurement of Newton's gravitational constant, $G = 6.67 \times 10^{-11}\text{m}^3\text{s}^{-2}\text{kg}^{-1}$, calculate the mass of the sun.

19. Use a graphing calculator or computer program to plot the curve defined by Eq. (72). (Pick representative values of the parameters a and e,) This is an example of a *polar plot*, in which the curve in a plane is defined by giving the radial distance r as a function of the angular position θ. Be sure to set the aspect ratio (= ratio of length scales on the horizontal and vertical axes) equal to 1.

20. Consider a particle M that moves on the xy plane. The polar coordinates (r, θ) and unit vectors $(\widehat{\mathbf{r}}, \widehat{\boldsymbol{\theta}})$ are defined in Fig. 15.

(a) Show that $x = r\cos\theta$ and $y = r\sin\theta$.

(b) Show that for a small displacement of M,

$$\Delta\widehat{\mathbf{r}} \approx \widehat{\boldsymbol{\theta}}\,\Delta\theta \qquad \text{and} \qquad \Delta\widehat{\boldsymbol{\theta}} \approx -\widehat{\mathbf{r}}\,\Delta\theta.$$

(Hint: See Fig. 7 but generalize it to include a radial displacement.)

(c) The position vector of M is $\mathbf{x} = r\widehat{\mathbf{r}}$, which has magnitude r and direction $\widehat{\mathbf{r}}$. In general, both r and $\widehat{\mathbf{r}}$ vary with time t, as the object moves. Show that

the velocity vector is

$$\mathbf{v} = \frac{dr}{dt}\,\widehat{\mathbf{r}} + r\frac{d\theta}{dt}\,\widehat{\boldsymbol{\theta}}.$$
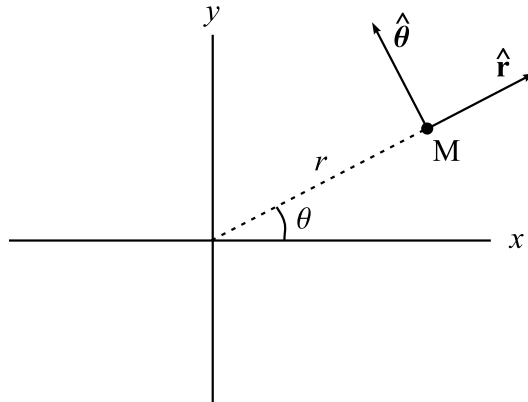

Figure 15: Exercise 20. Polar coordinates (r, θ) and unit vectors $\widehat{\mathbf{r}}$ and $\widehat{\boldsymbol{\theta}}$.

21. Derive Eq. (73) for the radial component a_r of the acceleration in polar coordinates. [Hint: Use the results of the previous exercise.]

22. Prove that the relation of parameters in (79) is true for a circular orbit. (For a circle, the eccentricity e is 0.)

23. Look up the orbital data—period T and semimajor axis a—for the planets. Calculate T^2/a^3 for all nine planets. Use the year (y) as the unit of time for T, and the astronomical unit (AU) as the unit of distance for a. Explain the values that you find for T^2/a^3.

24. TBA

25. **Reduced mass.** Suppose two masses, m_1 and m_2, exert equal but opposite forces on each other. Define the center of mass position $\mathbf{R}$ and relative vector $\mathbf{r}$ by

$$\mathbf{R} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2} \qquad \text{and} \qquad \mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2.$$

(Note that $\mathbf{r}$ is the vector from m_2 to m_1.)

(a) Show that $d^2\mathbf{R}/dt^2 = 0$, i.e., the center of mass point moves with constant velocity. (It could be at rest.)

(b) Show that

$$\mu \frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}(\mathbf{r})$$

where μ is the *reduced mass*, $m_1 m_2/(m_1 + m_2)$. Thus the two-body problem reduces to an equivalent one-body problem with the reduced mass.

(c) Show that Kepler's third law for the case of a circular orbit should properly be

$$T^2 = \frac{4\pi^2 r^3}{G(M + m)}$$

rather than (59). Why is (59) approximately correct?

General Exercises

26. ***Platform diving.*** A diver jumps off a 10 m platform. How many seconds does she have to do all her twists and flips before she enters the water? (Assume her initial upward velocity is 0.)

27. ***Conservation of energy.*** A rock falls from a cliff 100 m high, and air resistance can be neglected.

(a) Plot y (= height) versus t (= time).

(b) Plot v (= speed) versus t.

(c) Plot $v^2/2 + gy$ versus t. Describe the result in words.

28. ***Braking.*** (a) You are driving on the highway at 60 mph (= 88 ft/s). There is an accident ahead, so you brake hard, decelerating at $0.3\,g = 9.6\,\text{ft/s}^2$.

(a) How much time does it take to stop?

(b) How far will you travel before stopping?

(c) How far would you travel if your initial speed were 75 mph, assuming the same deceleration?

29. A ball rolling down an inclined plane has constant acceleration a. It is released from rest. U is a unit of length.

During the first second the ball travels a distance of 1 U on the inclined plane.

(a) How far does it travel during the second second?

(b) How far does it travel during the third second?

(c) How far does it travel during the tenth second?

(d) How far did it travel altogether after 10 seconds?

(e) What is a? (Express the answer in U/s².)

(f) Galileo made careful measurements of a ball rolling down an inclined plane, and discovered that the distance D is given by the equation $D = \frac{1}{2}at^2$. He observed that *the distances for fixed time intervals are in proportion to the sequence of odd integers.* Do your anwers agree?

30. A castle is 150 m distant from a catapult. The catapult projects a stone at 45 degrees above the horizontal. What initial speed v_0 is required to hit the castle?

(Hint: The initial velocity vector is $\mathbf{v}_0 = \widehat{\mathbf{i}}v_0 \cos 45 + \widehat{\mathbf{j}}v_0 \sin 45$; that is, $v_{0x} = v_{0y} = v_0/\sqrt{2}$.)

31. Parametric plots in Mathematica

A parametric plot is a kind of graph—a curve of y versus x where x and y are known as functions of an independent variable t called *the parameter*. To plot the curve specified by

$$x = f(t) \text{ and } y = g(t),$$

the Mathematica command is

```
ParametricPlot[{f[t],g[t]},{t,t1,t2},
  PlotRange->{{x1,x2},{y1,y2}},
  AspectRatio->r]
```

Here $\{$t1,t2$\}$ is the domain of t, and $\{$x1,x2$\}$ and $\{$y1,y2$\}$ are the ranges of x and y. To give the x and y axes equal scales, r should have the numerical value of (y2-y1)/(x2-x1).

Use Mathematica to make the parametric plots below. In each case name the curve that results.

(a) $x(t) = t, \quad y(t) = t - t^2$.

(b) $x(t) = t, \quad y(t) = 1/t$.

(c) $x(t) = \cos(2\pi t), \quad y(t) = \sin(2\pi t)$.

(d) $x(t) = 2\cos(2\pi t), \quad y(t) = 0.5\sin(2\pi t)$.

(e) $x(t) = \cos(2\pi t/3), \quad y(t) = \sin(2\pi t/7)$.

32. Baseball home run.
A slugger hits a ball. The speed of the ball as it leaves the bat is $v_0 = 100\,\text{mi/hr} = 147\,\text{ft/s}$. Suppose the initial direction is 45 degrees above the horizontal, and the initial height is 3 ft. The acceleration due to gravity is $32\,\text{ft/s}^2$.

(a) Plot y as a function of x, e.g., using Mathematica or a graphing calculator.

(b) When *precisely* does the ball hit the ground?

(c) Where *precisely* does the ball hit the ground?

(d) We have neglected air resistance. Is that a good approximation? Justify your answer.

33. Conservation of energy for a projectile

(a) Consider a projectile, moving under gravity but with negligible air resis-

tance, such as a shot put. Assume these initial values

$$x_0 = 0 \quad \text{and} \quad y_0 = 1.6\,\text{m},$$
$$v_{0x} = 10\,\text{m/s} \quad \text{and} \quad v_{0y} = 8\,\text{m/s}.$$

Use Mathematica or a graphing calculator to make plots of x versus t and y versus t. Show scales on the axes.

(b) Now plot the total energy (kinetic plus potential) versus t,

$$E(t) = \tfrac{1}{2}\left[\,v_x^2(t) + v_y^2(t)\,\right] + mgy(t)$$

for $m = 7\,\text{kg}$.

(c) Prove mathematically that E is a constant of the motion.

34. **The jumping squirrel.** A squirrel wants to jump from a point A on a branch of a tree to a point B on another branch. The horizontal distance from A to B is $x = 5\,\text{ft}$, and the vertical distance is $y = 4\,\text{ft}$. If the squirrel jumps with an initial speed of $20\,\text{ft/s}$, at what angle to the horizontal should it jump?

35. **Flight to Mars.** To send a satellite from Earth to Mars, a rocket must accelerate the satellite until it is in the correct elliptical orbit around the sun. The satellite does not travel to Mars under rocket power, because that would require more fuel than it could carry. It just moves on a Keplerian orbit under the influence of the sun's gravity.

The satellite orbit must have perihelion $r_- = R_E$ (= radius of Earth's orbit) and aphelion $r_+ = R_M$ (= radius of Mars's orbit) as shown in the figure. The planetary orbit radii are

$$R_E = 1.496 \times 10^{11}\,\text{m} \qquad \text{and} \qquad R_M = 2.280 \times 10^{11}\,\text{m}. \tag{82}$$

(a) What is the semimajor axis of the satellite's orbit?

(b) Calculate the time for the satellite's journey. Express the result in months and days, counting one month as 30 days.

36. **Parametric equations for a planetary orbit**

The sun is at the origin and the plane of the orbit has coordinates x and y. We can write parametric equations for the time t, and coordinates x and y,
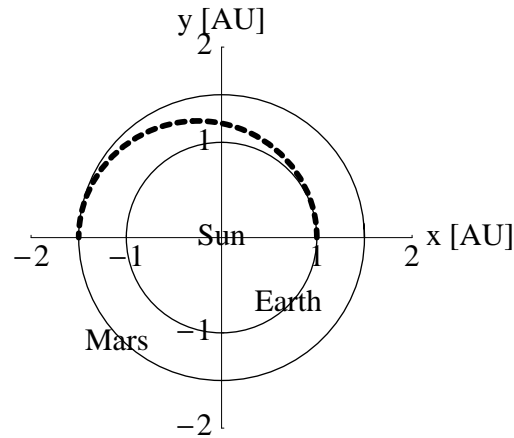
Figure 16: A journey to Mars.

in terms of an independent variable ψ:

$$
\begin{aligned}
t &= \frac{T}{2\pi}\left(\psi - \varepsilon \sin \psi\right) \\
x &= a\left(\cos \psi - \varepsilon\right) \\
y &= a\sqrt{1 - \varepsilon^2}\,\sin \psi
\end{aligned}
$$

The fixed parameters are $T =$ period of revolution, $a =$ semimajor axis, and $\varepsilon =$ eccentricity.

(a) The orbit parameters of Halley's comet are

$$a = 17.9\,\text{AU} \quad \text{and} \quad \varepsilon = 0.97.$$

Use Mathematica to make a parametric plot of the orbit of Halley's comet. (You only need the parametric equations for x and y, letting the variable ψ go from 0 to 2π for one revolution.)

(b) Calculate the perihelion distance. Express the result in AU.

(c) Calculate the aphelion distance. Express the result in AU. How does this compare to the radius of the orbit of Saturn, or Neptune?

(d) Calculate the period of revolution. Express the result in years.

37. **Parametric surfaces**

A parametric *curve* is a curve on a plane. The curve is specified by giving

coordinates x and y as functions of an independent parameter t.

A parametric *surface* is a surface in 3 dimensions. The surface is specified by giving coordinates x, y, and z as functions of 2 independent parameters u and v. That is, the parametric equations for a surface have the form

$$x = f(u, v), \qquad y = g(u, v), \qquad z = h(u, v).$$

As u and v vary over their domains, the points (x, y, z) cover the surface.

The Mathematica command for plotting a parametric surface is `ParametricPlot3D`. To make a graph of the surface, execute the command

```
ParametricPlot3D[{f[u,v],g[u,v],h[u,v]},
   {u,u1,u2},{v,v1,v2}]
```

In this command, (u_1, u_2) is the domain of u and (v_1, v_2) is the domain of v. Before giving the command you must define in Mathematica the functions `f[u,v]`, `g[u,v]`, `h[u,v]`. For example, for exercise (a) below you would define

```
f[u_,v_]:=Sin[u]Cos[v]
```

Make plots of the following parametric surfaces. In each case name the surface.

(a) For $0 \le u \le \pi$ and $0 \le v \le 2\pi$,

$$\begin{aligned} f(u,v) &= \sin u \cos v \\ g(u,v) &= \sin u \sin v \\ h(u,v) &= \cos u \end{aligned}$$

(b) For $0 \le u \le 2\pi$ and $-0.3 \le v \le 0.3$,

$$\begin{aligned} f(u,v) &= \cos u + v \cos(u/2) \cos u \\ g(u,v) &= \sin u + v \cos(u/2) \sin u \\ h(u,v) &= v \sin(u/2) \end{aligned}$$

(c) For $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$,

$$f(u,v) = 0.2(1 - v/(2\pi)) \cos(2v)(1 + \cos u) + 0.1 \cos(2v)$$

$$
\begin{aligned}
g(u,v) &= 0.2(1 - v/(2\pi))\sin(2v)(1 + \cos u) + 0.1\sin(2v) \\
h(u,v) &= 0.2(1 - v/(2\pi))\sin u + v/(2\pi)
\end{aligned}
$$

CONTENTS