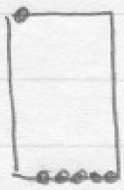
This was classical. What about quantum mechanical situations?

First assumption to go: indistinguishability



← is indistinguishable from any of the other 5 MAC's.. counts 1 rather than 6.

Now there are 20 distinguishable states, all equally likely.

There is no limit on the # particles in each state, so these cannot be electrons. They are Bosons and it was shown by Satyendra N. Bose in 1924 - nice story - and also by Einstein.

Called Bose-Einstein statistics.

Now calculate the number of states with $E=0$

MAC_1 probability is $\frac{1}{20}$ ← no microstates are distinguishable

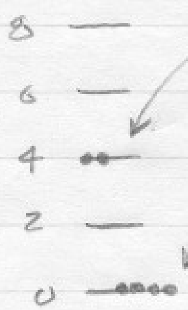
so, $n(E=0) = 5\left(\frac{1}{20}\right) = 0.25$

the other MAC states contribute $\frac{(5+4\times4+5\times3+5\times2+3\times1)}{20}$

$= 2.45$

Finally...

The spin $\frac{1}{2}$ distribution is quite different. Going back, what it would require is no state like

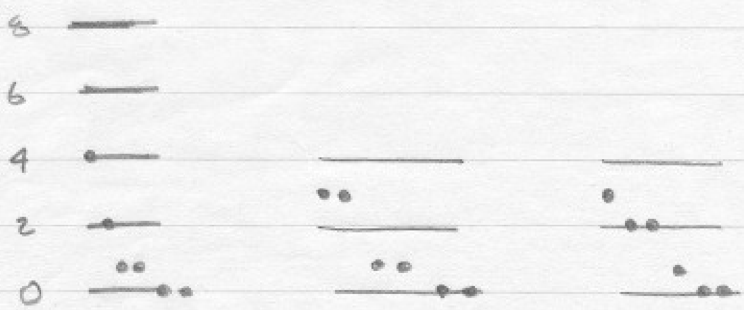


okay for electrons

violates the Pauli Exclusion principle

Still indistinguishable $\Rightarrow$ start with 20

But PE means only 3 MAC work.



Then $n_{FD}(E=0) = 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right) = 2.0$

"Fermi-Dirac"

table on next page

E	n_{MB}	n_{BE}	n_{FD}
0	2.31	2.45	2
1	1.54	1.55	1.07
2	0.98	0.90	1
3	0.59	0.45	1
4	0.33	0.30	0.33
5	0.16	0.15	0
6	0.07	0.10	0
7	0.02	0.05	0
8	0.005	0.05	0

not so different, but generally more in the lower E states.

Rethink our derivation of MB

Consider 2 energy states E_1 and E_2 occupied by an average n_1 and n_2 particles.

rate that n_1's become n_2's : $R_{1\to 2}$

n_2's n_1's : $R_{2\to 1}$

Both are rates per particle – probabilities per second per particle.

total rate of $1 \to 2$ transitions: $n_1 R_{1\to 2}$

At Equilibrium

$$n_1 R_{1\to 2} = n_2 R_{2\to 1}$$

$$\frac{n_1}{n_2} = \frac{R_{2\to 1}}{R_{1\to 2}}$$

$n_1 = A e^{-E_1/kT}$ etc. so,

$$\frac{R_{2\to 1}}{R_{1\to 2}} = \frac{e^{-E_1/kT}}{e^{-E_2/kT}}$$

Now consider Bosons.

$$n_1 R^B_{1\to 2} = n_2 R^B_{2\to 1}$$

Recall the discussion of Ψ_A and Ψ_S

$\uparrow$

appropriate for
electrons – spin 1/2

Ψ_S is appropriate for the other kind of spin-objects.
Bosons .. spin 0, 1/2 ... etc.

Consider 2 identical bosons. The symmetric wave function:

$$\Psi_S = \sqrt{1/2}\ \left[\Psi_\alpha(1)\Psi_\beta(2) + \Psi_\beta(1)\Psi_\alpha(2) \right]$$

normalization, since are normalized.
α, β summarize all identifiers for a "state"

Now .. put both bosons into the same state, $\alpha = \beta$

$$\Psi_S = \sqrt{1/2}\ \left[\Psi_\beta(1)\Psi_\beta(2) + \Psi_\beta(1)\Psi_\beta(2) \right]$$

$$= \sqrt{2}\ \Psi_\beta(1)\Psi_\beta(2)$$

So, $$\Psi_S^*\Psi_S = 2\ \Psi_\beta^*(1)\Psi_\beta^*(2)\Psi_\beta(1)\Psi_\beta(2)$$

Suppose the quantum requirement of indistinguishability had not been applied? We would have stuck with

$$\Psi_T = \Psi_\alpha(1)\Psi_\beta(2) \qquad \text{.. already normalized.}$$

if same state:

$$\Psi_T^*\Psi_T = \Psi_\beta^*(1)\Psi_\beta^*(2)\Psi_\beta(1)\Psi_\beta(2)$$

$$P(\psi_s) = 2\,P(\psi_T)$$

Probability of having 2 quantum mechanical bosons in same state is twice that for 2 classical bosons.

Suppose 3 particles?

$$\psi_s = \sqrt{1/6}\,\big[\,\psi_\alpha(1)\psi_\beta(2)\psi_\gamma(3) + \psi_\beta(1)\psi_\gamma(2)\psi_\alpha(3)$$

$$+\ \psi_\gamma(1)\psi_\alpha(2)\psi_\beta(3) + \psi_\gamma(1)\psi_\beta(2)\psi_\alpha(3)$$

$$+\ \psi_\beta(1)\psi_\alpha(2)\psi_\gamma(3) + \psi_\alpha(1)\psi_\gamma(2)\psi_\beta(3)\,\big]$$

6 terms. = 3! terms.

Setting same state: $\alpha = \beta = \gamma$:

$$\psi_s = \underset{3!}{\sqrt{1/3}} \cdot \overset{3!}{6}\ \psi_\beta(1)\psi_\beta(2)\psi_\beta(3)$$

$$\psi_s^*\psi_s = \left(\frac{1}{3!}\right)(3!)^2\ \underbrace{\psi_\beta^*(1)\psi_\beta^*(2)\psi_\beta^*(3)\psi_\beta(1)\psi_\beta(2)\psi_\beta(3)}_{\substack{\text{classical result} \\ \psi_T^*\psi_T}}$$

$$\psi_s^*\psi_s = 3!\ \psi_T^*\psi_T$$

generalizable

→ an enhancement effect for bosons of $n!$

Consider an empty state, add a boson →

P_1 = probability that the boson will land in that state.

adding additional bosons... each would have the same probability — classically.

$$P_n^c = (P_1)^n$$

But, quantum mechanically, the enhancement

$$P_n^B = n!\, P_n^c = n!\,(P_1)^n$$

The probability that there would be $n+1$

$$P_{n+1}^B = (n+1)!\; P_{n+1}$$

since

$$(n+1)! = (n+1)n! \quad \Rightarrow \quad P_{n+1}^B = (n+1)n!\; P_{n+1}$$

and

$$P_{n+1} = (P_1)^{n+1} = (P_1)^n P_1 = P_n^c P_1$$

$$P_{n+1}^B = (n+1)\underline{n!\; P_n^c} P_1$$

or.

$$P_{n+1}^B = (n+1)\, P_1\, P_n^B$$

prob that there are n B in state already

prob of classical of adding 1

If there are already n bosons in a quantum state, the prob of one more joining them is $(n+1)\times$ classical now, rates. for bosons.

Equilibrium says,

$$n_1 R^B_{1\to 2} = n_2 R^B_{2\to 1}$$

relate to the classical case.

$$R^B_{1\to 2} = (1+n_2) R_{1\to 2}$$ already assuming that there are n_2 in 2

$$R^B_{2\to 1} = (1+n_1) R_{2\to 1}$$

so,

$$n_1 (1+n_2) R_{1\to 2} = n_2 (1+n_1) R_{2\to 1}$$

$$\frac{n_1(1+n_2)}{n_2(1+n_1)} = \frac{R_{2\to 1}}{R_{1\to 2}} = \frac{e^{-E_1/kT}}{e^{-E_2/kT}}$$

$$\frac{n_1}{1+n_1} e^{E_1/kT} = \frac{n_2}{1+n_2} e^{E_2/kT} = f(T)$$

$$\frac{n_1}{1+n_1} e^{E_1/kT} = f$$

$$\frac{n_1}{1+n_1} = f \cdot e^{-E_1/kT}$$ let $f = e^{-\alpha}$ $\alpha(T)$

$$= e^{-\alpha} e^{-E_1/kT}$$

$$n_1 = n_1 e^{-(\alpha + E_1/kT)} + e^{-(\alpha + E_1/kT)}$$

$$n_1\left[1-e^{-(\alpha+E_1/kT)}\right] = e^{-(\alpha+E_1/kT)}$$

so
$$n_1 = \frac{e^{-(\alpha+E_1/kT)}}{1-e^{-(\alpha+E_1/kT)}}$$

$$n_1 = \frac{1}{e^{\alpha}e^{E_1/kT}-1}$$

same result
for n_2

$$n_B(E) = \frac{1}{e^{\alpha}e^{E/kT}-1} = \frac{1}{B_2\,e^{E/kT}-1} = F_{BE}$$

book

notice

Bose-Einstein
Distribution.

$$F_{MB} = A\,e^{-E/kT} = \frac{1}{A'\,e^{E/kT}}$$

For electrons... the Pauli Exclusion Principle introduces an inhibition factor

$$R^F_{1\to2} = (1-n_2)\,R_{1\to2}$$

$$R^F_{2\to1} = (1-n_1)\,R_{2\to1}$$

... same calculation
$$n_F(E) = \frac{1}{e^{\alpha}e^{E/kT}+1}$$

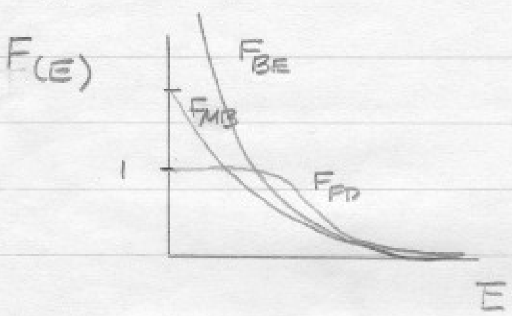
or $$F_{FD} = \frac{1}{B_1 e^{E/kT} + 1}$$

Fermi-Dirac Distribution

So, here we are — probability distributions —

		LIMITING cases E/kT tiny	E/kT large
$F_{MB} =$	$\frac{1}{A' e^{E/kT}}$	constant	tiny
$F_{BE} =$	$\frac{1}{B_2 e^{E/kT} - 1}$	∞	tiny
$F_{FD} =$	$\frac{1}{B_1 e^{E/kT} + 1}$	smaller constant	tiny

They have the following relative shapes at some high temperature



like the little 4 particle example

Remember what these mean

$$n(E)\,dE = g(E)\,F_X(E)\,dE$$

can determine B_1 and B_2 by

$$\left(\frac{N}{V}\right)_B = \int_0^\infty g(E)\,F_{BE}(E)\,dE$$

$$\left(\frac{N}{V}\right)_F = \int_0^\infty g(E)\,F_{FD}(E)\,dE$$