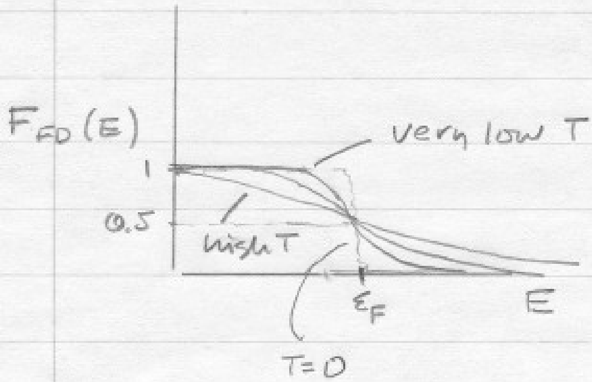
Fermi-Dirac Distributions.




The $e^{\alpha}$ term in $\quad n_{FD}(E) = \dfrac{1}{e^{\alpha} e^{E/kT} + 1}$

is special... written as

$$e^{\alpha} = e^{-\varepsilon_F/kT}$$ called the "Fermi Energy"

$$n_{FD}(E) = \frac{1}{e^{(E-\varepsilon_F)/kT} + 1}$$

notice, when $E = \varepsilon_F$ $\quad n_{FD}(\varepsilon_F) = \dfrac{1}{1+1} = 0.5$

half states have $E > \varepsilon_F$

Notice also, that when

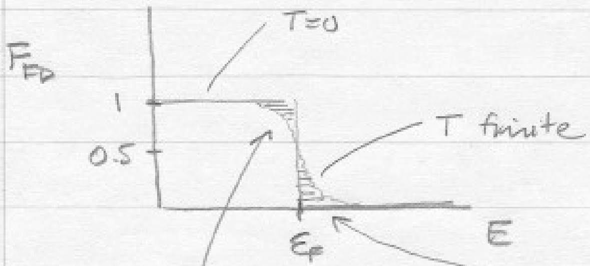
$E \ll \varepsilon_F$ $\quad n_{FD}(E) \simeq \dfrac{1}{e^{-\varepsilon_F/kT} + 1} \sim 1$

now such states contain 1 fermion per quantum state, per E

} occupation space → MB

$\varepsilon_F$ } about half full

} filled from lowest E up.




states near $\varepsilon_F$ depopulate and start to occupy

states with E in just a bit larger.

So, with most energy levels occupied for $E < \varepsilon_F$ and a few electrons outside of that -- this is a model of

<u>a metal</u> — the occupied states are like the confinement in a 3d lattice.

→ use the ∞ well in 3d as a model.

$$E = \frac{h^2}{8mL^2}\left(n_1^2 + n_2^2 + n_3^2\right)$$

Notice practically.... $\varepsilon_F = kT_F$ -- $T_F \sim 8.12 \times 10^4$ K Cu

$6.36 \times 10^4$ K Ag

so, $\varepsilon_F$ is high for metals.

$\varepsilon_F = 7$ eV Cu

5.6 eV Ag.

Example.. what's a typical velocity for an electron in Copper at the Fermi "surface"?

$$\tfrac{1}{2} m_e v_F^2 = E_F$$

$$v_F^2 = \frac{2E_F}{m_e} = \frac{2E_F c^2}{m_e c^2}$$

$$\frac{v_F^2}{c^2} = \frac{(2)(7\,eV)}{511\times10^3\,eV}$$

$$\frac{v_F}{c} = 0.005 \quad \Rightarrow \quad 1.6\times10^6\ m/s \quad !$$

So, back

write $\quad n_1^2 + n_2^2 + n_3^2 \equiv \eta^2$

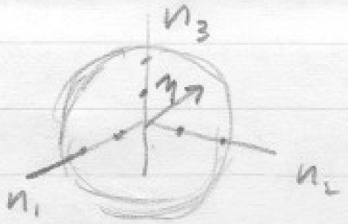
$$E_1 \equiv \frac{h^2}{8mL^2} \qquad \text{(remember)}$$

so $\quad E = \eta^2 E_1$

want to calculate $g(E)$ in order to get $n(E)$

\# states per unit energy.

Think of a "sphere" in quantum number space.




at some $\eta$, the # would be related to the volume in this space

$$N_\eta = \left(\tfrac{4}{3}\pi\eta^3\right)\left(\tfrac{1}{8}\right)(2)$$

for FD!
2 spin states per energy.

not whole
sphere since $n_i > 0$

$$N_\eta = \frac{\pi\eta^3}{3}$$

since $\quad E = \eta^2 E_1$

$$\eta = \sqrt{\frac{E}{E_1}} = \sqrt[3]{\frac{3N_\eta}{\pi}}$$

so, $\quad \eta^3 = \left(\frac{E}{E_1}\right)^{3/2} = \frac{3N_\eta}{\pi}$

$$N_\eta = \frac{1}{3}\pi\left(\frac{E}{E_1}\right)^{3/2}$$

$N(E)$:
# states at $\eta$

$g(E)$ = # states per unit energy.

$$g(E) = \frac{dN_\eta}{dE} = \frac{\pi}{2}(E_1)^{-3/2}\underline{E^{1/2}}$$

consider $T=0$ $\qquad n(E)\,dE = g(E)\,F_{FD}(E, T=0)\,dE$

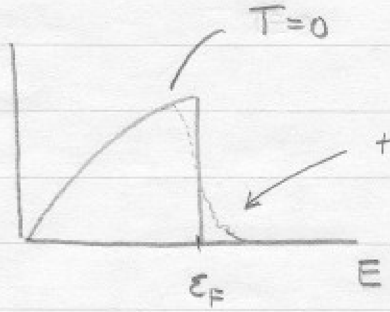
$$F_{FD}(E, T=0) = 1 \quad E < \varepsilon_F \qquad = 0 \quad E > \varepsilon_F$$

$$n(E)\,dE = \frac{\pi}{2}(E_1)^{-3/2}E^{1/2}\,dE \qquad E < \varepsilon_F$$

$$= 0 \qquad E > \varepsilon_F$$

For non-zero T

$$n(E)\,dE = \frac{\pi}{2}(E_1)^{-3/2}\frac{E^{1/2}\,dE}{e^{(E-E_F)/kT}+1}$$




If a model for real metals, these are conduction electron states.
— a <u>gas</u> of electrons of high speeds.

This explained an historical problem. If electrons in a metal were classical,

$$U = N(3/2\,kT) = \frac{3}{2}RT$$

so $$C_{el}^{c} = \frac{\partial U}{\partial T} = \frac{3}{2}R$$

which is large — comparable to the contribution to C from the lattice vibrations

— but experiment did not show this

But — the # electrons available as a "gas" are tiny. ~ % or so.

Bose - Einstein Distributions
For photons in Black body —

$$n(E)\,dE = g(E)\,F_{BE}(E)\,dE$$

The energy density    energy per unit volume -

$$u(E)\,dE = E\,n(E)\,dE$$

$$= \frac{g(E)\,E\,dE}{e^{E/kT}-1}$$

$B_2 \to 1$ .. photons are not normalized, walls absorb and emit.

Need $g(E)$ again. Can't use the massive particle in a box for massless electrons -

Recast it as momentum.

$$E_\eta = \frac{h^2}{8\cancel{m}L^2}\eta^2 = \frac{p^2}{2\cancel{m}}$$

$$p^2 = \frac{h^2}{4L^2}\eta^2$$

$$p = \sqrt{p_x^2 + p_y^2 + p_z^2} = \frac{h}{2L}\eta$$

so $$E = pc = \frac{hc}{2L}\eta$$

$$N_\eta = \left(\frac{4}{3}\pi\eta^3\right)\left(\frac{1}{\cancel{8}}\right)\cancel{(2)}$$

accidentally like FD..
2 polarization states for $\gamma$'s

$$N_\eta = \frac{\pi}{3}\,\frac{8L^3}{h^3c^3}\,E^3$$

$$g(E) = \frac{dN_\eta}{dE} = \frac{8\pi L^3}{h^3c^3}E^2$$

$$n(E)\,dE = g(E)\,F_{BE}(E)\,dE$$

$$= \frac{8\pi L^3}{h^3c^3} E^2 \frac{1}{e^{E/kT}-1}$$

$$u(E)\,dE = \frac{E\,n(E)\,dE}{L^3} = \frac{8\pi}{h^3c^3} E^3 \frac{1}{e^{E/kT}-1}\,dE$$

$$E = \frac{hc}{\lambda}$$

$$|dE| = \frac{hc}{\lambda^2}\,d\lambda$$

$$u(\lambda, T)\,d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT}-1}\,d\lambda$$

the Planck formula! once one multiplies by c/4 to go from energy density to intensity

→ Einstein did this.

Einstein solid.

Einstein treated the vibrations of crystals (as a result of heating) by quantizing the oscillations themselves not the atoms.

→ give rise to excitations ("quasiparticles") of the lattice which exist only inside of the crystal — Bose-Einstein "particles" called phonons.

Assumed all phonons have same frequency, so no density of states to be concerned with

Remember $E = \hbar\omega(n + 1/2)$, now each $n$ is creation of a new phonon.

So, presume a mole ⇒ $N_A$ atoms ⇒ $3N_A$ phonons each with energy $\hbar\omega$.

The probability distribution for having phonons is

$$\frac{1}{e^{\hbar\omega/kT} - 1}$$ so the energy per mole

is

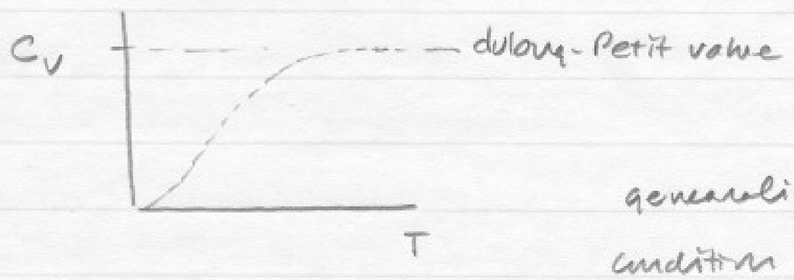
$$E = 3N_A \hbar\omega \frac{1}{e^{\hbar\omega/kT} - 1}$$

and the heat capacity

$$C = \frac{dE}{dT} = 3N_A\hbar\omega \frac{(e^{\hbar\omega/kT})(\hbar\omega/kT^2)}{(e^{\hbar\omega/kT}-1)^2}$$

$$= 3R\left(\frac{\hbar\omega}{kT}\right)^2 \frac{e^{\hbar\omega/kT}}{(e^{\hbar\omega/kT}-1)^2}$$

which reproduces the low T behavior well




generalizing the condition that all phonons have same energy leads to the Debye model — slight differences

Strange behavior —

Recall for the FD gas. $E_1 = \frac{h^2}{8mL^2}$

and

$$g_{FD}(E) = \frac{\pi}{2}(E_1)^{-3/2} E^{1/2}$$

$$= \frac{\pi}{2}\left(\frac{8mL^2}{h^2}\right)^{3/2} E^{1/2}$$

$$g_{FD}(E) = \frac{\pi V}{h^3} 4(2m)^{3/2} E^{1/2}$$

this will work for BE also, but for the factor of 2 introduced because of the Pauli Exclusion Principle

$$g_{BE}(E) = \frac{2\pi V}{h^3} (2M)^{3/2} E^{1/2}$$

not electrons any more.
some object which we'll presume
behaves like a BE particle.

$$n(E) = g_{BE}(E) F_{BE}$$

$$= \frac{2\pi V}{h^3} (2M)^{3/2} E^{1/2} \frac{1}{B_2 e^{E/kT} - 1}$$

$$N = \int_0^\infty n(E)\, dE = \frac{2\pi V}{h^3} (2M)^{3/2} \int_0^\infty \frac{E^{1/2}\, dE}{B_2 e^{E/kT} - 1}$$

$B_2$ is usually written as another exponential (like with the Fermi Energy)

$$B_2 = e^{-\mu/kT} > 0 \qquad \text{So} \int \frac{E^{1/2}\, dE}{e^{(E-\mu)/kT} - 1}$$

so as T gets small, $\mu(T)$ has to change to keep N positive $\mu(T)$ has to increase toward 0

At $\mu(T_c) = 0$ and $B_2 = 1$, which is its minimum value.

set $B = 1$

$$N = \frac{2\pi V}{h^3} (2m)^{3/2} k^{3/2} T_c^{3/2} \underbrace{\int_0^\infty \frac{x^{1/2}\,dx}{e^x - 1}}_{2.315}$$

$$N \leq \frac{2\pi V}{h^3} (2mk)^{3/2} T_c^{3/2} (2.315)$$

Below $T_c$ there are problems.

Einstein treated the states having $T < T_c$ differently from those above

$$\underset{He_{II}}{T} < T_c < \underset{He_{I}}{T} < T_v \quad \text{gas.}$$

something different! – a liquid

B.E. liquid satisfying N above

a phase transition of an unusual kind.

$^4$He consists of 2 protons and 2 neutrons. The spins are such that He nucleus is a Boson.

Putting in the $m_{^4He} = 2m_p + 2m_n$

$$T_c = \frac{h^2}{2mk} \left[ \frac{N}{V} \frac{1}{2\pi (2.315)} \right]^{2/3} \sim 3\,K$$

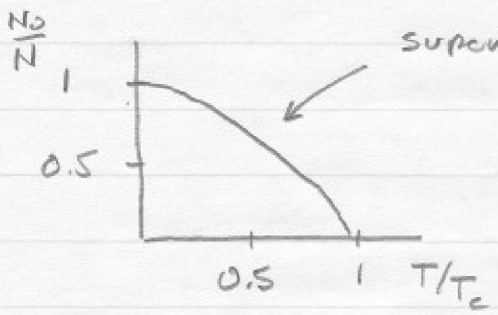
(actually 2.17 K)

Below this temperature, the liquid consists of 2 kinds of liquids

$$N = N_0 + N_n$$

$N_0$: ground state $He_{II}$

$N_n$: normal $He_I$




the ground state FILLS

called "Bose-Einstein Condensation".

$N_0$ atoms in the liquid are all in the same quantum state

LHe does this - visually impressive is L $^{87}Rb$ -

Superfluids - zero viscosity.

macroscopic quantum mechanical states!