The applications of Newton’s laws kept coming

Thomas Jefferson
- even designed a plow contour to maximize the overturn of earth for minimum force

Bernoulli’s were a family of always amazing, sometimes eminent mathematicians
- but dysfunctional beyond belief

Daniel was the most accomplished, certainly as a practical developer of both Leibniz and Newton’s work

It was he who uncovered the use of Newton’s laws, when combined with the conservation of vis viva to fully explain fluid dynamics

\[ P + \rho v^2 = \text{constant} \]

Another conservation law, explaining the flow of fluids through varying sized pipes - motivated by Harvey’s model of the circulation system

- high pressure low velocity
- low pressure high velocity
- high pressure low velocity

All published in *Hydrodynamica*, in 1738, 40, 42 along with an atomic explanation of Boyle’s Law
THE DEVIL OF GASES IS IN THE DETAILS...

"KINETIC THEORY OF GASES"

very old, very cool calculation... soon after N.!

IMAGINE A VOLUME FILLED WITH MOLECULES:

- identical, mass m
- point-like $\Rightarrow$ no size $\Rightarrow$ no interactions
- numerous enough to ignore fluctuations $\sim \frac{1}{V N}$

... "IDEAL GAS!"

\[ \frac{1}{1} \]

\[ \frac{1}{L} \]

\[ \frac{1}{2} \]

\[ \frac{1}{3} \]

a cube of molecules
recoils elastically from walls

$\Delta t =$ average time between $x = 0$ and $x = L$ collisions
\[ \Delta t = \frac{2L}{u_x} \]

Momentum is transferred to wall - elastic

so,

\[ \Delta P_x = 2m |u_x| \]

Rate at which momentum is transferred:

\[ \frac{\Delta P_x}{\Delta t} = \frac{2m|u_x|}{2L} = \frac{mu_x^2}{L} = \langle F_x \rangle \]

by one molecule

Total force applied to \( x = 0 \) or \( x = L \) walls:

\[ \langle F \rangle = \frac{Nm u_x^2}{L} \]

The pressure:

\[ P = \frac{\langle F \rangle}{A} = \frac{Nm u_x^2}{L^2 \cdot L} = \frac{Nm u_x^2}{V} \]

Assumes all molecules have the same speeds -- use averages

\[ u_x^2 \rightarrow \langle u_x^2 \rangle \]

\[ P = \frac{Nm \langle u_x^2 \rangle}{V} \]
X isn't special -- $u$ and $y$ are, on average, the same.

\[ \langle u^2 \rangle + \langle v^2 \rangle + \langle y^2 \rangle = \langle v^2 \rangle \]

so,

\[ \langle u^2 \rangle = \frac{1}{3} \langle v^2 \rangle \]

so,

\[ P = N m \frac{\langle v^2 \rangle}{3 v} \]

or,

\[ P V = N m \frac{\langle u^2 \rangle}{3} \]

BOYLE worked with the chemist Robert Hooke and found that at a constant $T$:

\[ P V = \text{constant} \]

Boyle's Law

"springiness of air"

A century later Jacques Charles found

\[ \frac{V}{T} = \text{constant} \]

-- at constant Pressure

Charles's Law
THESE GO TOGETHER...

\[ PV = nRT \]

or \[ PV = NkT \]

**Ideal Gas Law**

"Equation of State"

\[ n = \text{# moles} \]

\[ R = \text{Universal Gas Constant} = 8.314510 \times 10^{-23} \text{ mol}^{-1} \]

\[ k = \text{Boltzmann's Constant} = 1.38066 \times 10^{-23} \text{ J/K} \]

\[ N = \text{# molecules} \]

HERE'S THE MAGIC:

\[ PV = Nm\langle u^2 \rangle = nkT \]

\[ \Rightarrow \quad hT = m\langle u^2 \rangle \]

\[ hT = \frac{2}{3} \left( \frac{1}{2} m\langle u^2 \rangle \right) = \frac{2}{3} \langle k \rangle \]

Gas Temperature can be thought of as the average kinetic energy of the molecules of a gas.

\[ \Delta U = N(\Delta k) \quad U = NK \quad \text{total internal energy} \]

so:

\[ PV = \frac{2}{3} U \]
\[ \langle K \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} kT \]

\[ \langle v^2 \rangle = \frac{3 kT}{m} \Rightarrow \langle v_x^2 \rangle = \frac{kT}{m} \]

**FLIP IT**

\[ \text{mean rms} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3 kT}{m}} \]

**a measure of the average speed of a molecule from**

Is it its temperature?

What's the average speed of air molecules at 20°C?

Mass of N\(_2\) is 28 amu.

28 x 28

\[ m = 28 \times 1.66 \times 10^{-27} \text{ kg} \]

\[ = 4.648 \times 10^{-26} \text{ kg}. \]

\[ \text{mean rms} = \sqrt{\frac{3 kT}{m}} \]

\[ = \sqrt{\frac{(3)(1.38 \times 10^{-23} \text{ J/K})(293 \text{ K})}{4.65 \times 10^{-26} \text{ kg}}} \]

\[ \text{mean rms} = 510 \text{ m/s} \]

Ideal... what about my perfume?
SO... MORE TO THE VELOCITIES OF
GAS MOLECULES THAN JUST AVERAGES...
- they are distributed over wide ranges
MAXWELL FIGURED THIS OUT...

... introducing probability into physics...
( described in Chapter 9 - here's a sketch, we'll come back)

LET'S THINK ABOUT A

A. NUMBER DISTRIBUTION

B. PROBABILITY DISTRIBUTION
consider I94 to Chicago...

what's the # distribution of cars at a given speed... guess something like:

between, say
B.C. and K200

watch for 30 minutes
and find

the total # of cars is $N = 5 + 10 + 20 + 30 + 40 + 30 + 10$

$N = 145$

the probability that a car might be going at 55 mph --

$$f(55) \approx \frac{30}{145} = 0.21 \approx 21\%$$

if the measured distribution is representative of any time...
The number density is defined so that

\[ \int_0^{100} n(v) dv = N \]

so, since a histogram with finite \( \Delta x \), the integral here is a sum...

\[ \sum_{i=1}^{10} n_i \Delta v = N \]

\[ \text{the bin width} = 10 \text{ mph} \]

\[ \sum_{i=1}^{10} n_i \Delta v = (0 + 0 + 5 + 1 + 2 + 3 + 4 + 3 + 1) \times 10 = 145 \]

\( n(v) dv \) is the number of cars between \( v \) and \( v + dv \) — here \( v + \Delta v \)

\( f(v) dv \) is the probability distribution of finding a car with velocity between \( v \) and \( v + dv \).
WHAT MAXWELL DID WAS CALCULATE

THE NUMBER DENSITY (OR PROBABILITY)

FOR MOLECULAR SPEEDS IN A GAS

using simple ideas...

Again, assume that $v_x, v_y, v_z$ are independent...

so probabilities multiply

(we'll do this later... here are the results)

The probability distribution for a single component of velocity, say $v_x$

$$N \ f(v_x) \ dv_x = N \left( \frac{m}{2\pi kT} \right)^{\frac{3}{2}} e^{-\frac{mv_x^2}{2kT}} \ dv_x$$

Gaussian

all he did was insist that

$$\langle v_x^2 \rangle = \frac{kT}{m}$$
SPEEDS -- ANOTHER STORY

\[ n(v) \, dv = 4\pi N \left( \frac{m}{2\pi kT} \right)^{3/2} v^2 e^{-\frac{mv^2}{2kT}} \]

called the Maxwell Speed Distribution

- You give me the constituent \((m)\)
- You give me the temperature \((T)\)
- I'll tell you how many molecules there are at any speed \((v)\)

\[
\frac{n(v)}{N} \quad 0.8 \quad e \quad 0.4
\]

\[ \langle v \rangle = 1.1284 \cdot \bar{v}_m \]
\[ v_{rms} = 1.2268 \cdot \bar{v}_m \]

\[ \bar{v}_m = \sqrt{\frac{2kT}{m}} \]
Room temperature $N_2$ molecules
Here, the escape velocity is about 10,000 m/s and that still above that amount is enough, over 1.5 billion years, to have lost all of the original primordial hydrogen in our atmosphere.
Room temperature H$_2$ molecules
notice the scale difference from before
IDEAL GAS...

A remarkably robust model for many kinds of gases:

- large N
- point objects $\rightarrow$ no size
- identical molecules / bits...
- no forces among particles, just the walls.

$\rightarrow$ most gases approach this for small $P$!

The Ideal Gas Law is remarkably consistent with our simple Bernoulli-inspired mechanical model.

\[ PV = nRT \]

or

\[ PV = Nk_BT \]

GO BACK TO P-V DIAGRAMS, WORK ETAL.

for the specific Ideal Gas model.
ISOThermal EXPANSION...

Using \( PV = nRT = \text{constant} \quad \text{(constant } T) \)

\[
P \quad \frac{V}{i} \quad \text{a particular shape in } P-V
\]

for an "ISOThERM"

same Temperature along the curve

so, \( P = \frac{nRT}{V} \) is the curve

\[
P = \frac{\text{constant}}{V}
\]

hyperbola

\[
P \quad \frac{1}{V} \quad \text{each a different temperature}
\]

High \( T \) out
Low \( T \) in

WHAT ABOUT WORK DONE BY AN IDEAL GAS SYSTEM?

\[
\Delta W = \int P \, dV
\]

\[
= \int_{V_i}^{V_f} \frac{nRT}{V} \, dV = nRT \ln \frac{V_f}{V_i}
\]

\( V_f > V_i \) volume bigger \( \rightarrow \) expansion \( \Rightarrow \Delta W > 0 \)

\( V_f < V_i \) volume smaller \( \rightarrow \) compression \( \Rightarrow \Delta W < 0 \)