PHY481: Electrostatics

Semester plans

Introductory E&M review (1)

Plan of attack

- Compassionate review
  - 1 week reviewing Intro E&M concepts, including brief descriptions and solving "familiar" problems.
  - Homework assignment: to derive E & V for typical charge distributions.
- Followed by the typical course content
  - Developed advanced mathematics and techniques
  - Full description of each topic in Electrostatics, using advanced mathematics, and solving problems with a large range of difficulty
  - Exams: ~40% at an Intro E&M level, ~60% with focus on advanced techniques.
  - I expect that you can, at a minimum, do the Intro problems!

Properties of classical electric charge

- Electric Charge
  - Property of matter associated with the electromagnetic force
  - Magnitude quantized in units (or 1/3) of electron charge *e*

$$e = 1.6 \times 10^{-19} \text{ C}$$

  - Two signs + (proton) and – (electron), also neutral (neutron)
- All matter begins as a collection of neutral atoms
  - Electrons can move from one object to another.
  - To make an object + (–), remove (add) electrons.
  - An object with charge +q, implies a net charge –q elsewhere.
- Charge densities ρ, σ, λ with simple space dependences

Volume	Surface	Line
$dq = \rho dV$	$dq = \sigma dS$	$dq = \lambda d\ell$

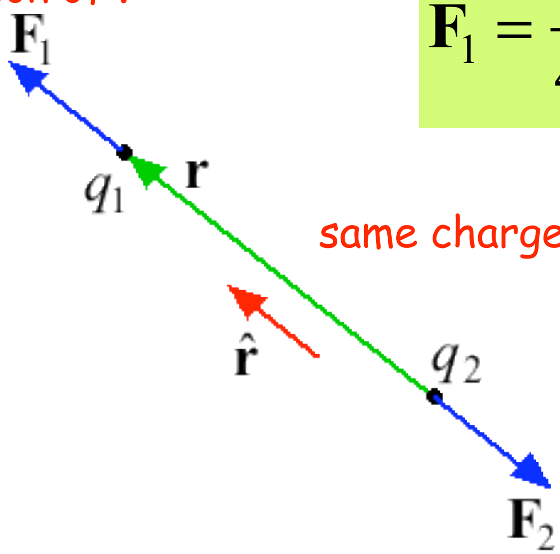
Force between charges

- Force between two charges, Coulomb's Law:

Force on 1 is
in direction of **r**

$$\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}$$

Will need new
notation later!



same charge sign

$$\frac{1}{4\pi\varepsilon_0} = 9.0\times 10^9\ \frac{\mathrm{Nm}^2}{\mathrm{C}^2}$$

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} \text{ points from } q_2 \text{ to } q_1$$

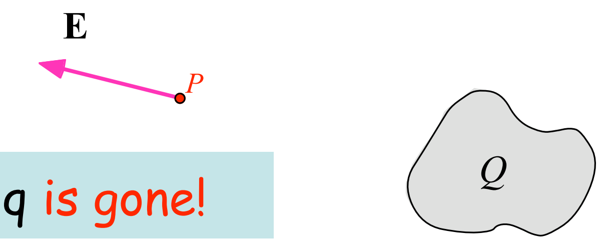
q_1 and q_2 carry charge sign

- Superposition principle
  - Force on charge q from charges q_1,q_2, ... q_k is the vector sum of forces between q and each of the charges.
  - No interference between action of the charges

The electric field

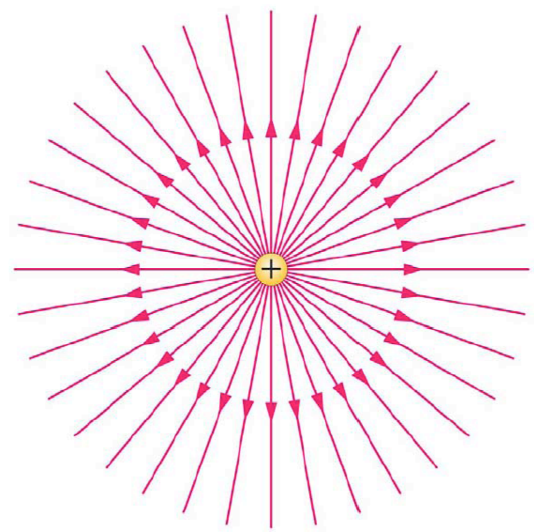
- Charge creates an electric field
  - A very small positive charge q placed at a point P experiences a force **F** from a collection of charge Q (seems positive).
  - The electric field **E** at the point P is defined as

$$\mathbf{E} = \frac{\mathbf{F}}{q}$$



q is gone!

- Electric field lines
  - begin on + charge and end on – charge
  - direction of **E** is along field lines
  - **E** field lines do not cross
  - density of lines is α to field magnitude



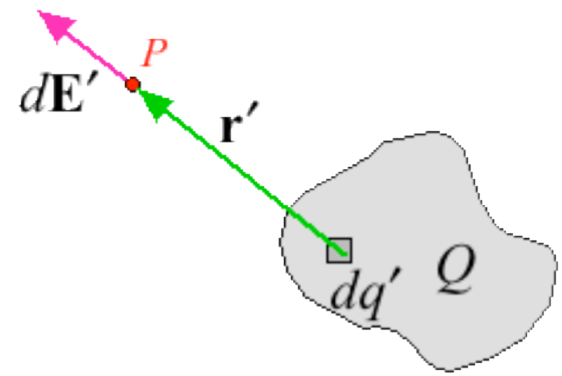
positive point charge

Electric fields from charge distributions

- Integration over charge distributions

Will need new notation later!

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{dq'}{r'^2}\hat{\mathbf{r}}'$$



$$dq' = \rho dV'$$
$$Q = \int \rho dV'$$

- E-fields of simple charge distributions & density
  - Sphere, cylinder, box - volume charge density ρ
  - Sphere, cylinder, box, sheet - surface charge density σ
  - Thin line, ring - linear charge density λ
  - Sheets and lines may be of infinite extent

Coordinate systems

- Unit vectors, differential line and space elements
  - Cartesian
    $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$

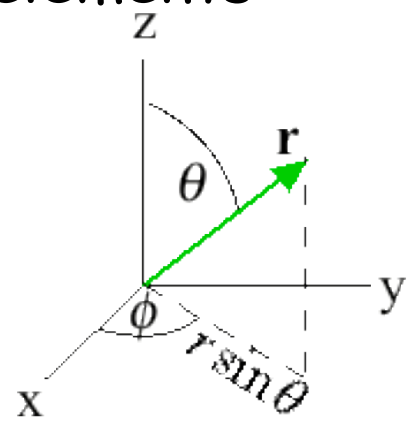
    $$dx$$
    $$dV = dx\,dy\,dz$$

  - Cylindrical
    $\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{k}}$

    $$dr \text{ (radial) or } rd\phi \text{ (ring)}$$
    $$dV = rdrd\phi dz$$

  - Spherical
    $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$

    $$dr \text{ (radial)}, \quad rd\theta \text{ (polar)}, \quad r\sin\theta d\phi \text{ (ring)}$$
    $$dV = r^2 dr \sin\theta d\theta d\phi$$



  - Symmetry used to avoid angular complications
  - Radial and angular unit vectors needed later.
- Warm up! From the above, determine the volume and surface area of a cylinder & sphere of radius R.

Electric field from point charges

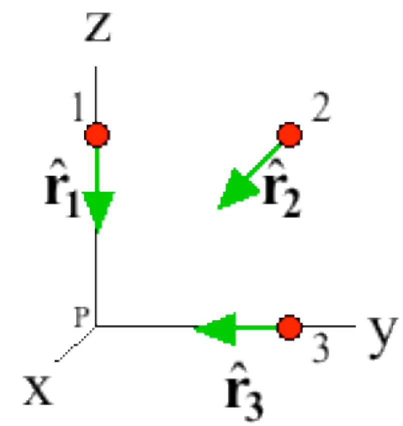
- The electric field $\mathbf{E}_p$ generated by point charges at point P is the vector sum of $\mathbf{E}_i$ from each charge:

$$\mathbf{E}_p = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{\mathbf{r}}_i$$

  - Find electric field at the origin due to the three charges $q_{1\text{-}3}$ on corners of a square with side *a*.

$$\mathbf{E}_p = \frac{1}{4\pi\varepsilon_0}\left[-\frac{q_1}{a^2}\hat{\mathbf{k}} - \frac{q_2}{2a^2}\left(\frac{\hat{\mathbf{j}}+\hat{\mathbf{k}}}{\sqrt{2}}\right) - \frac{q_3}{a^2}\hat{\mathbf{j}}\right]$$

$$= \frac{-1}{4\pi\varepsilon_0 a^2}\left[\left(q_1 + \tfrac{\sqrt{2}}{4}q_2\right)\hat{\mathbf{k}} + \left(q_3 + \tfrac{\sqrt{2}}{4}q_2\right)\hat{\mathbf{j}}\right]$$

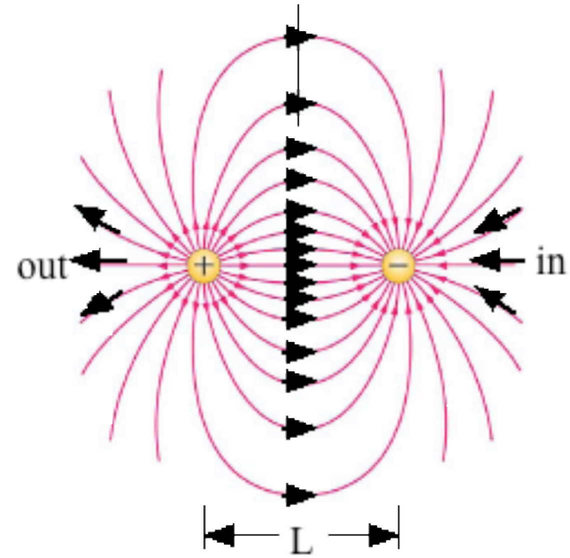


Dipole field on the bisector

- Field line's direction is out of +q and into −q
  - Definition of dipole moment vector with charges on the x-axis

$$\mathbf{p} = q\mathbf{L} = -qL\hat{\mathbf{i}}$$

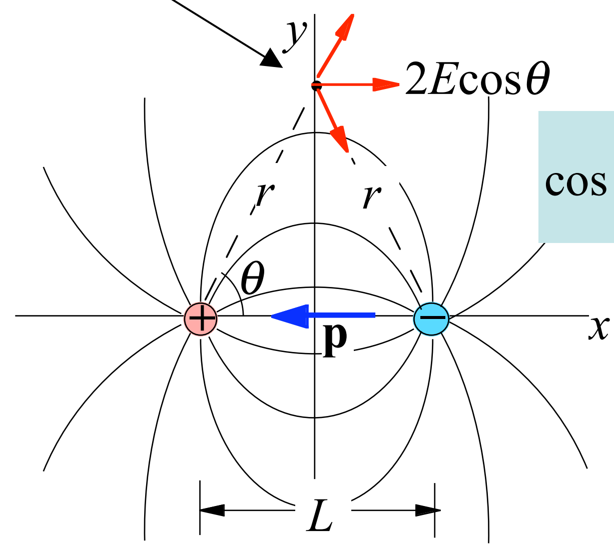
  - On the bisector, the vertical components cancel, horizontal components add.



$$\mathbf{E} = \frac{2}{4\pi\varepsilon_0}\frac{q}{r^2}\cos\theta\,\hat{\mathbf{i}} = \frac{1}{4\pi\varepsilon_0}\frac{qL}{r^3}\hat{\mathbf{i}}$$

$$= \frac{-\mathbf{p}}{4\pi\varepsilon_0 r^3}$$

Note minus sign



$$\cos\theta = \frac{L/2}{r}$$

$$r \approx y$$

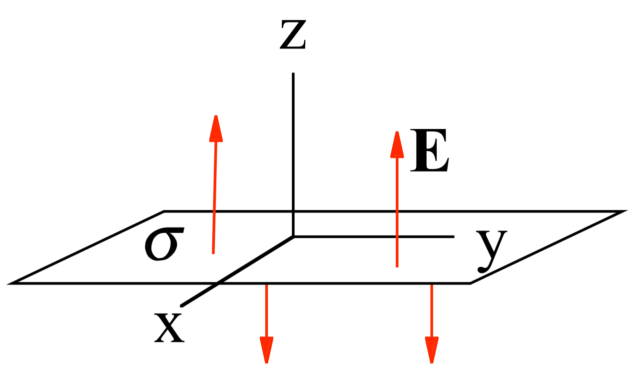
$$\mathbf{E} = \frac{-\mathbf{p}}{4\pi\varepsilon_0 y^3}$$

Uniformly charged infinite plane

- For an infinite horizontal plane the only reasonable direction for the electric field **E** is vertical.
- Electric field can be determined by integrating over the charge distribution (try it yourself). It is not too surprising that the field is the same at all distances above the plane.

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}} \text{ (above)}$$

$$\mathbf{E} = -\frac{\sigma}{2\varepsilon_0}\hat{\mathbf{k}} \text{ (below)}$$

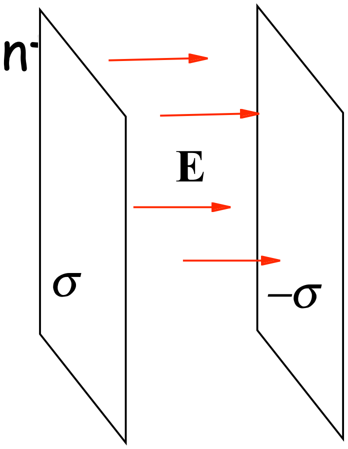


- The change in the electric field going from below to above

$$\Delta E = \frac{\sigma}{\varepsilon_0}$$

Parallel charge sheets

- Two infinite sheets of charge are separated by a constant distance d. One sheet has a charge density +σ and the other a charge density −σ.
  - Outside, the electric fields point in opposite directions
  - Between the sheets the electric fields point in the same direction.



$$\mathbf{E}_{outside} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{i}} + \frac{\sigma}{2\varepsilon_0}(-\hat{\mathbf{i}}) = 0$$

Outside plates field is zero

$$\mathbf{E}_{inside} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{i}}$$

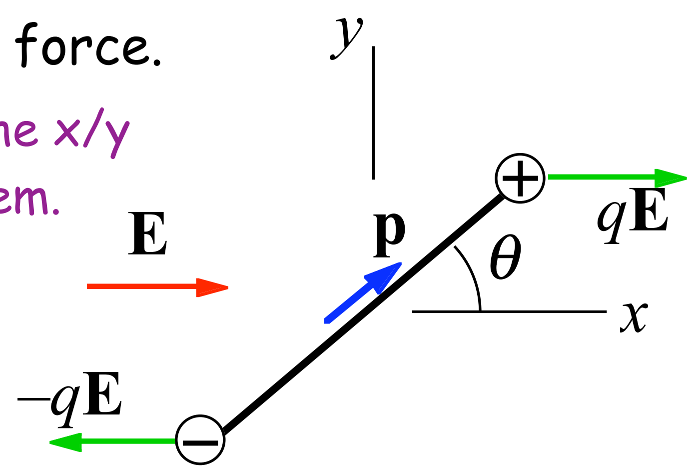
Field between the plates

- Uniform electric field E, applies a constant force on a small particle with charge q and mass m.

$$\mathbf{F} = q\mathbf{E} \text{ and } \mathbf{a} = \frac{\mathbf{F}}{m} = \frac{q}{m}\mathbf{E}$$

Torque on a small electric dipole

- An electric dipole **p** in a uniform electric field **E** experiences a net torque $\mathbf{N}$ and no net force.
  - Choose coordinates where **p** and **E** lie in the x/y plane. **p** and **E** have an angle θ between them.



$$\mathbf{N} = \mathbf{p} \times \mathbf{E} = pE\sin\theta(-\hat{\mathbf{k}})$$

- In addition to a torque, an electric field E with a divergence will generate, a net force **F** on an electric dipole, **p** :

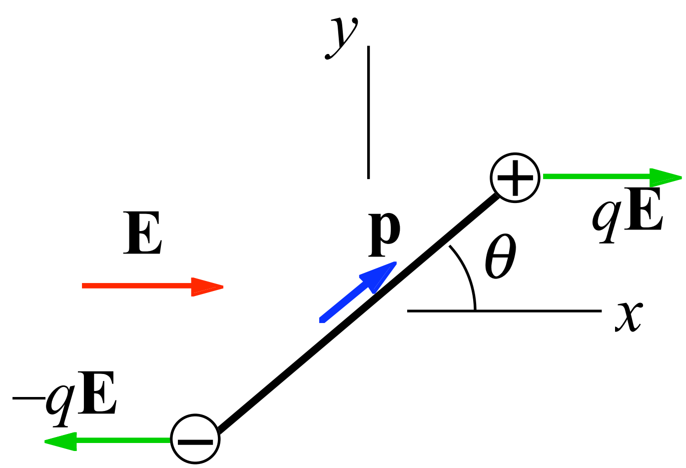
Cartesian coordinates

$$\mathbf{F} = p_i \frac{\partial E_j}{\partial x_i} \hat{\mathbf{e}}_j$$

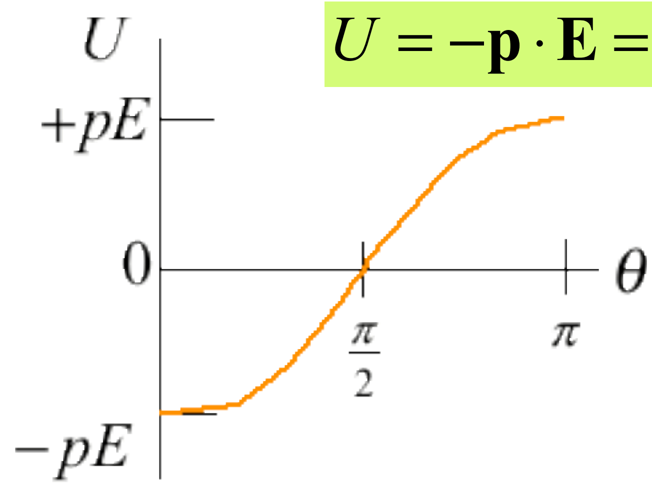
General expression needs operators to be covered later

Energy of dipole in electric field

- Potential energy *U* of the electric dipole **p** in uniform electric field **E**:

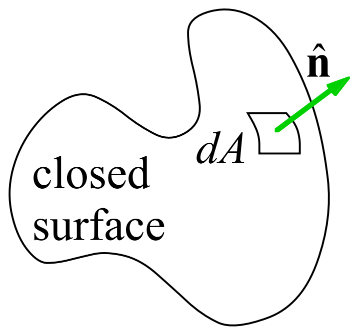


$$U = -\mathbf{p}\cdot\mathbf{E} = -pE\cos\theta$$

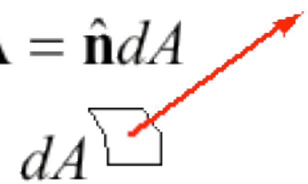


Gauss's Law

- Electric field passing through a closed (mathematical) surface
  - A surface enclosing NO net charge has a zero net field leaving or entering the surface.
  - A surface enclosing a positive (negative) charge has a net field leaving (entering) the surface proportional to the enclosed charge.



$$d\mathbf{A} = \hat{\mathbf{n}}\,dA$$



$$\int_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{encl}}{\varepsilon_0}$$

General expression for Gauss's Law

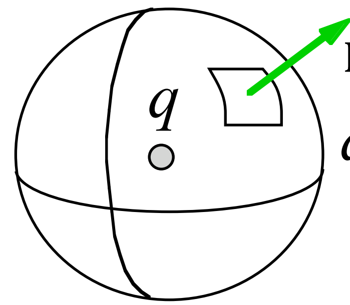
- For symmetric charge distributions, pick an enclosing surface where **E** and *d***A** are everywhere parallel to each other.

Coulomb's Law <---> Gauss's Law

- For symmetric charge distributions, pick enclosing surfaces, so that **E** and *d***A** are are parallel to each other.
  - For a point charge at the origin, use a spherical surface, radius *R*, centered on the charge (makes direction of normal = radial)

Electric field at surface

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 R^2}\hat{\mathbf{r}}$$



$$\hat{\mathbf{n}} = \hat{\mathbf{r}}$$

$$dA = R^2 \sin\theta\, d\theta\, d\phi$$

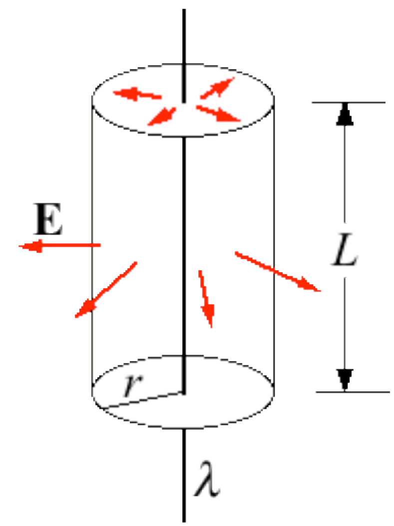
Evaluate Gauss's Integral

$$\int_S \mathbf{E}\cdot d\mathbf{A} = \frac{q}{4\pi\varepsilon_0 R^2}\int_0^{\pi} R^2 \sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{q}{\varepsilon_0}$$

- This is a "proof" that Gauss's law follows directly from the Coulomb Force Law for point charges, and their derived electric fields.

Field of a line of charge - use Gauss's Law

- Consider an infinitely long line of charge with linear charge density λ, and a cylindrical gaussian surface.
  - The electric field is parallel to the surface at the top and bottom of the cylinder, **E**·d**A** is zero.
  - The electric field is perpendicular to the surface and therefore parallel to the surface normal.



$$\int_S \mathbf{E}\cdot d\mathbf{A} = Er\int_0^{2\pi} d\phi \int_0^L dz$$

$$E2\pi rL = \frac{q_{encl}}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}; \qquad \mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{\mathbf{r}}$$

$$q_{encl} = \lambda L$$