PHY481: Electromagnetism

Vector tools

Sorry, no office hours today

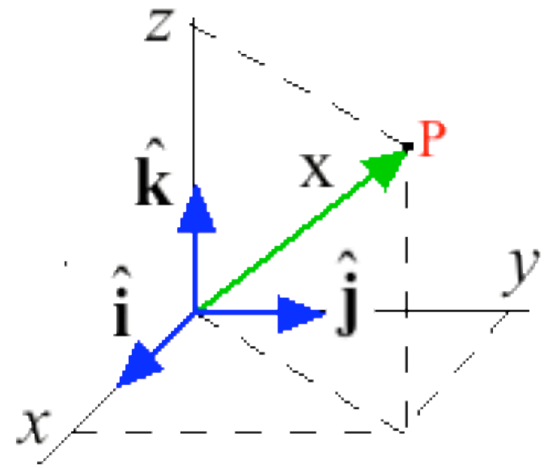
I've got to catch a plane

for a meeting in Italy

Definitions

- Cartesian coordinates
  - Vector **x** is defined relative to the origin of primary coordinate system (x,y,z)
  - In Cartesian coordinates

$$\mathbf{x} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$$



- Arbitrary rotation of the coordinate system
  - changes vector direction but not the magnitude

$$\mathbf{x}' = x'\,\hat{\mathbf{i}}' + y'\,\hat{\mathbf{j}}' + z'\,\hat{\mathbf{k}}'$$

$$|\mathbf{x}'| = |\mathbf{x}| = \sqrt{x^2 + y^2 + z^2}$$

- Example: rotation about the z-axis by angle ϕ

$$\hat{\mathbf{i}}' = \hat{\mathbf{i}}\cos\phi + \hat{\mathbf{j}}\sin\phi$$
$$\hat{\mathbf{j}}' = -\hat{\mathbf{i}}\sin\phi + \hat{\mathbf{j}}\cos\phi$$
$$\hat{\mathbf{k}}' = \hat{\mathbf{k}}$$

$$x' = \mathbf{x}\cdot\hat{\mathbf{i}}' = x\cos\phi + y\sin\phi$$
$$y' = \mathbf{x}\cdot\hat{\mathbf{j}}' = -x\sin\phi + y\cos\phi$$
$$z' = z$$

Rotation matrix

- A rotation can be expressed as a 3x3 unitary matrix R
  - Rotation of coordinates

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

  - Rotation of vector components

$$\begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix} = R \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

  - Rotation Matrix is Unitary
    - Maintains vector magnitude
    - Transpose = Inverse

$$R^T_{ij} = R_{ji} = R^{-1}_{ij}$$

  - Rotation matrix properties
    - R = BCD (3 rotations)
    - Rotations do not commute

Euler angle rotations

$$B = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ rotation about z

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$ rotation about x′

$$D = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ rotation about z′

Vector multiplication

- Vector addition and multiplication by a constant is OK
- Scalar (or dot) product

$$\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y + A_z B_z = \sum_{i=1}^{3} A_i B_i$$

(1,2,3) = (x,y,z)

- Einstein summation convention (repeated indices)

$$\mathbf{A}\cdot\mathbf{B} = A_i B_i \quad = A_1 B_1 + A_2 B_2 + A_3 B_3$$

Σ is superfluous

- Cross product (new techniques)

$$\mathbf{C} = \mathbf{A}\times\mathbf{B}$$

$$C_i = A_j B_k - A_k B_j$$

i,j,k are cyclic permutations of 1,2,3

$$C_x = A_y B_z - A_z B_y$$

$$C_1 = A_2 B_3 - A_3 B_2$$

(1, 2, 3)

$$C_y = A_z B_x - A_x B_z$$

$$C_2 = A_3 B_1 - A_1 B_3$$

(1, 2, 3) → (2, 3, 1)

$$C_z = A_x B_y - A_y B_x$$

$$C_3 = A_1 B_2 - A_2 B_1$$

(2, 3, 1) → (3, 1, 2)

Economy of notation we will NEED

- Levi-Chivita anti-symmetric 3x3x3 tensor ε_{ijk}

Learn to love it ! $(\mathbf{A}\times\mathbf{B})_i = \varepsilon_{ijk}A_jB_k$ $\quad i, j, k$ = (1, 2 or 3)

If any 2 indices are the same $\varepsilon_{ijk} = 0$ 21 of 27 elements are 0 !

Permutations of 123 — 6 non-zero elements

Cyclic = even # of pair-wise $\quad \varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = +1$

Non-cyclic = odd # of pair-wise $\quad \varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1$

$$(\mathbf{A}\times\mathbf{B})_1 = \varepsilon_{123}A_2B_3 + \varepsilon_{132}A_3B_2 = A_2B_3 - A_3B_2$$

Verify the remainder for yourself

- Levi-Chivita tensor product

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

sum over k, k = 1, 2, 3

Kronecker delta

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Vector identities (Levi-Chivita tensor)

- Prove **A x (B x C) = B(A·C) – C(A ·B)**

Cross product using Levi-Chivita tensor: $(\mathbf{A}\times\mathbf{B})_i = \varepsilon_{ijk}A_jB_k$

Levi-Chivita tensor product: $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

Start proof here: $\mathbf{B}\times\mathbf{C} = \mathbf{D}$

i 'th component $[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})]_i = (\mathbf{A}\times\mathbf{D})_i = \varepsilon_{ijk}A_jD_k$

$$D_k = (\mathbf{B}\times\mathbf{C})_k = \varepsilon_{klm}B_lC_m$$

need indices *l* and *m*

$$[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})]_i = \varepsilon_{ijk}\varepsilon_{klm}A_jB_lC_m$$

$$= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)A_jB_lC_m$$

$$= A_jB_iC_j - A_jB_jC_i$$

$$A_jB_j = \mathbf{A}\cdot\mathbf{B}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

Vector operators - Gradient (Cartesian)

- Gradient operates on a scalar function V (potential)

Gives a vector in the direction of the maximum change in V

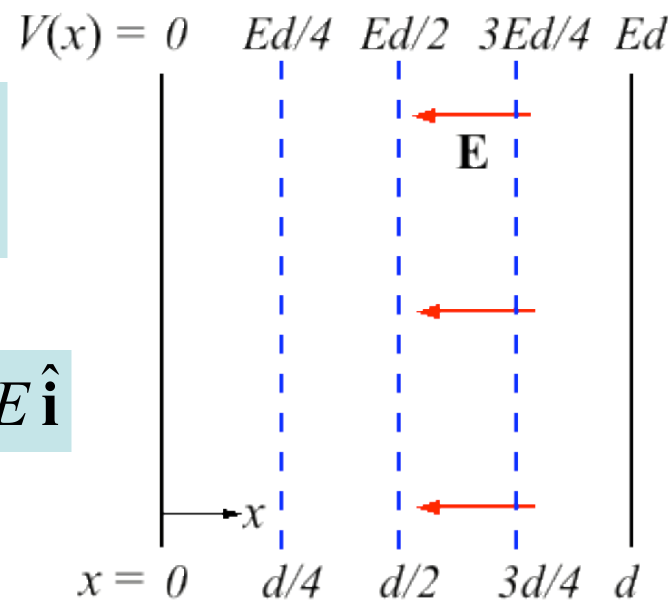
$$\nabla V(\mathbf{x}) = \frac{\partial V}{\partial x_i}\hat{\mathbf{e}}_i = \frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}$$

$$\nabla = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}$$

$$\hat{\mathbf{e}}_1 = \hat{\mathbf{i}} \quad \hat{\mathbf{e}}_2 = \hat{\mathbf{j}} \quad \hat{\mathbf{e}}_3 = \hat{\mathbf{k}}$$

- Example: parallel plate capacitor

$$V(\mathbf{x}) = V(x) = Ex$$



Maximum change in V is in the +x direction

$$\nabla V(\mathbf{x}) = \frac{\partial V}{\partial x}\hat{\mathbf{i}} = E\,\hat{\mathbf{i}}$$

Electric field **E** points in the direction opposite to maximum change in V, -x direction

$$\mathbf{E}(\mathbf{x}) = -\nabla V(\mathbf{x}) = -E\,\hat{\mathbf{i}}$$

Vector operators - Divergence (Cartesian)

- Divergence of a vector function is a scalar

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{\partial E_i}{\partial x_i} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

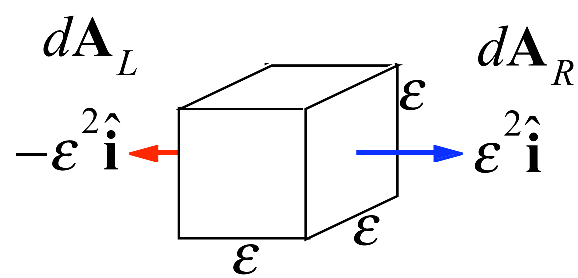
Spreading of **E** at **x**.
≠ 0 if charge is at **x**

Del operator

$$\nabla = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}$$

- Coordinate independent definition

tiny box centered on **x**



$$\oint_S \mathbf{E}(\mathbf{x}) \cdot d\mathbf{A} = \sum_{i=1}^{3} \left[E_i(\mathbf{x} + \tfrac{\varepsilon}{2}\hat{\mathbf{e}}_i)\varepsilon^2 - E_i(\mathbf{x} - \tfrac{\varepsilon}{2}\hat{\mathbf{e}}_i)\varepsilon^2 \right]$$

$$= \sum_{i=1}^{3} \left[\frac{E_i(\mathbf{x} + \tfrac{\varepsilon}{2}\hat{\mathbf{e}}_i) - E_i(\mathbf{x} - \tfrac{\varepsilon}{2}\hat{\mathbf{e}}_i)}{\varepsilon} \right] \varepsilon^3$$

$$= \sum_{i=1}^{3} \left[\frac{\partial E_i}{\partial x_i} \right] \varepsilon^3 = [\nabla \cdot \mathbf{E}(\mathbf{x})]\varepsilon^3$$

Divergence definition

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \lim_{V \to 0} \frac{1}{V} \oint_S \mathbf{E}(\mathbf{x}) \cdot d\mathbf{A}$$

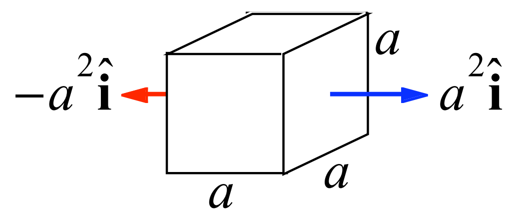
Differential form of Gauss's Law

- Gauss's Law in terms of divergence of **E**

Gauss's Law

$$\oint_S \mathbf{E}(\mathbf{x}) \cdot d\mathbf{A} = \frac{q_{encl}}{\varepsilon_0}$$

Box centered on **x**



$$\frac{1}{V}\oint_S \mathbf{E}(\mathbf{x}) \cdot d\mathbf{A} = \frac{1}{V}\frac{q_{encl}}{\varepsilon_0}$$

$$\rho = \frac{q_{encl}}{V}; \quad V = a^3$$

$$\lim_{V\to 0}\frac{1}{V}\oint_S \mathbf{E}(\mathbf{x}) \cdot d\mathbf{A} = \lim_{V\to 0}\frac{q_{encl}}{V}\frac{1}{\varepsilon_0}$$

see previous slide

Gauss's Law equivalent

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \lim_{V\to 0}\frac{1}{V}\oint_S \mathbf{E}(\mathbf{x}) \cdot d\mathbf{A}$$

Vector operators - Curl (Cartesian)

- Curl of a vector function is another vector

$$[\nabla \times \mathbf{B}(\mathbf{x})]_i = \varepsilon_{ijk} \frac{\partial B_k}{\partial x_j}$$

"Circulation" of E around a loop

$$(\nabla \times \mathbf{B}(\mathbf{x}))_1 = \frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3}$$

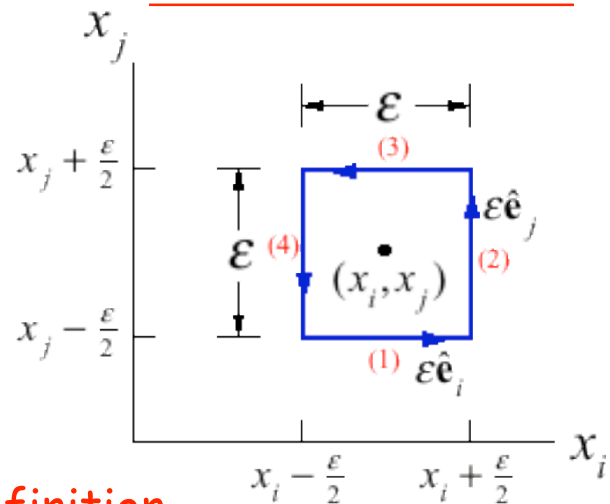
Verify the remainder

- Coordinate independent definition

$$\oint_{P_k} \mathbf{B}(\mathbf{x}) \cdot d\ell = \Big[\overset{(1)}{B_i(\mathbf{x} - \tfrac{\varepsilon}{2}\hat{\mathbf{e}}_j)\varepsilon} - \overset{(3)}{B_i(\mathbf{x} + \tfrac{\varepsilon}{2}\hat{\mathbf{e}}_j)\varepsilon} \Big] + \Big[\overset{(2)}{B_j(\mathbf{x} + \tfrac{\varepsilon}{2}\hat{\mathbf{e}}_i)\varepsilon} - \overset{(4)}{B_j(\mathbf{x} - \tfrac{\varepsilon}{2}\hat{\mathbf{e}}_i)\varepsilon} \Big]$$

$$= \left[\frac{\partial B_j}{\partial x_i} - \frac{\partial B_i}{\partial x_j} \right] \varepsilon^2 = [\nabla \times \mathbf{B}(\mathbf{x})]_k \, \varepsilon^2$$

Path centered on **x**



Curl definition

$$\hat{\mathbf{n}} \cdot [\nabla \times \mathbf{B}(\mathbf{x})] = \lim_{A \to 0} \frac{1}{A} \oint_C \mathbf{B}(\mathbf{x}) \cdot d\ell$$

Cartesian vector operators (Einstein notation)

Summary of earlier slide 4

Dot Product

$$\mathbf{A}\cdot\mathbf{B} = A_i B_i$$

Levi-Chivita Tensor

$$\varepsilon_{ijk}$$

Cross Product

$$(\mathbf{A}\times\mathbf{B})_i = \varepsilon_{ijk} A_j B_k$$

Permutations of 123

6 non-zero elements

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Non-cyclic = odd # of pair-wise

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Tensor product

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Kronecker delta

$$\delta_{ij} = \begin{cases} 1\,, & i = j \\ 0\,, & i \neq j \end{cases}$$

Vector differential operators in Einstein notation

Del

$$\nabla = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}$$

Gradient

$$\nabla V(\mathbf{x}) = \frac{\partial V}{\partial x_i}\hat{\mathbf{e}}_i$$

Divergence

$$\nabla\cdot\mathbf{E}(\mathbf{x}) = \frac{\partial E_i}{\partial x_i}$$

Curl

$$[\nabla\times\mathbf{B}(\mathbf{x})]_i = \varepsilon_{ijk}\frac{\partial B_k}{\partial x_j}$$

$$\hat{\mathbf{e}}_1 = \hat{\mathbf{i}},\ \hat{\mathbf{e}}_2 = \hat{\mathbf{j}},\ \hat{\mathbf{e}}_3 = \hat{\mathbf{k}}$$

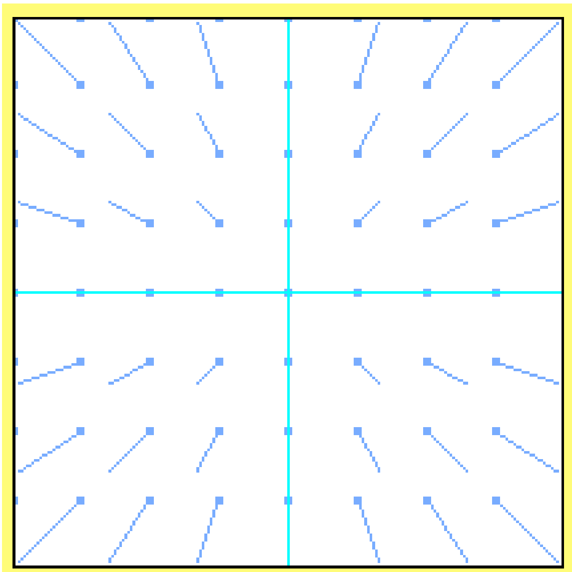
$$\nabla\cdot\nabla V = \nabla^2 V$$

Laplacian of scalar function *V*

Physical interpretation of vector operators

- Characterize "flow" of field [area displayed (±3m, ±3m)]

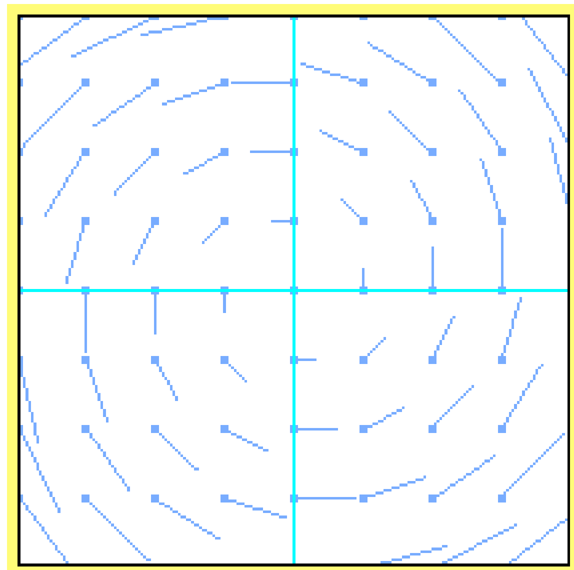
$$\mathbf{E}(\mathbf{x}) = \left(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}\right)\ \mathrm{V/m}^2$$



$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{x})}{\varepsilon_0} = 2\ \mathrm{V/m}^2$$

$$\nabla \times \mathbf{E} = 0$$ E is "irrotational"

$$\mathbf{B}(\mathbf{x}) = \left(-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}\right)\ \mathrm{T/m}^2$$

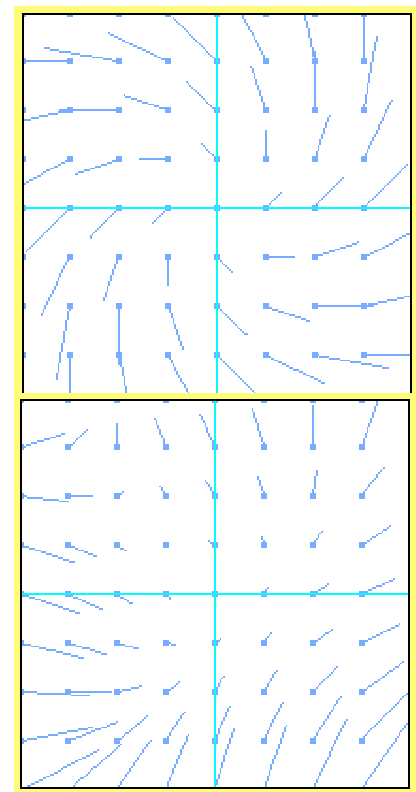


$$\nabla \cdot \mathbf{B} = 0$$ B is "solenoidal"

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}(\mathbf{x}) = 2\ \mathrm{T/m}^2$$

$$\nabla \times \mathbf{C} \neq 0$$

$$\nabla \cdot \mathbf{C} \neq 0$$



(see http://www.math.gatech.edu/~carlen/2507/notes/vectorCalc/)