
PHY481: Electromagnetism

Del identities & integral theorems

Cartesian vector operators (Einstein notation)

Dot Product

$$\mathbf{A} \cdot \mathbf{B} = A_i B_i$$

Levi-Chivita Tensor

$$\epsilon_{ijk}$$

Permutations of 123

Cyclic = even # of pair-wise

Non-cyclic = odd # of pair-wise

Cross Product

$$(\mathbf{A} \times \mathbf{B})_i = \epsilon_{ijk} A_j B_k$$

6 non-zero elements

$$\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = +1$$

$$\epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1$$

Tensor product

$$\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

Kronecker delta

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Del

$$\nabla = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}$$

Gradient

$$\nabla V(\mathbf{x}) = \frac{\partial V}{\partial x_i} \hat{\mathbf{e}}_i$$

Divergence

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{\partial E_i}{\partial x_i}$$

Curl

$$[\nabla \times \mathbf{B}(\mathbf{x})]_i = \epsilon_{ijk} \frac{\partial B_k}{\partial x_j}$$

$$\hat{\mathbf{e}}_1 = \hat{\mathbf{i}}, \hat{\mathbf{e}}_2 = \hat{\mathbf{j}}, \hat{\mathbf{e}}_3 = \hat{\mathbf{k}}$$

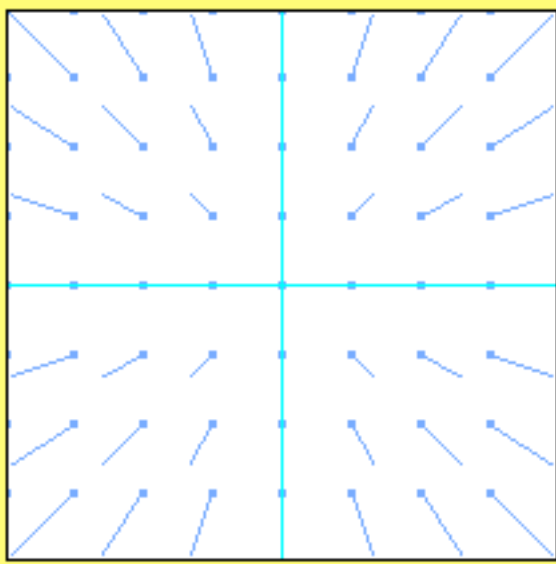
$$\nabla \cdot \nabla V = \nabla^2 V$$

Laplacian of scalar function V

Physical interpretation of vector operators

- Characterize "flow" of field [area displayed ($\pm 3\text{m}$, $\pm 3\text{m}$)]

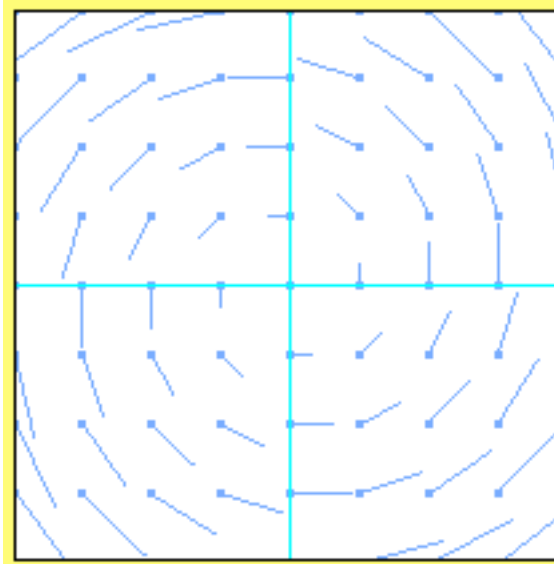
$$\mathbf{E}(\mathbf{x}) = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}}) \text{ V/m}^2$$



$$\nabla \cdot \mathbf{E} = \frac{\rho(\mathbf{x})}{\epsilon_0} = 2 \text{ V/m}^2$$

$$\nabla \times \mathbf{E} = 0 \quad \mathbf{E} \text{ is "irrotational"}$$

$$\mathbf{B}(\mathbf{x}) = (-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}) \text{ T/m}^2$$

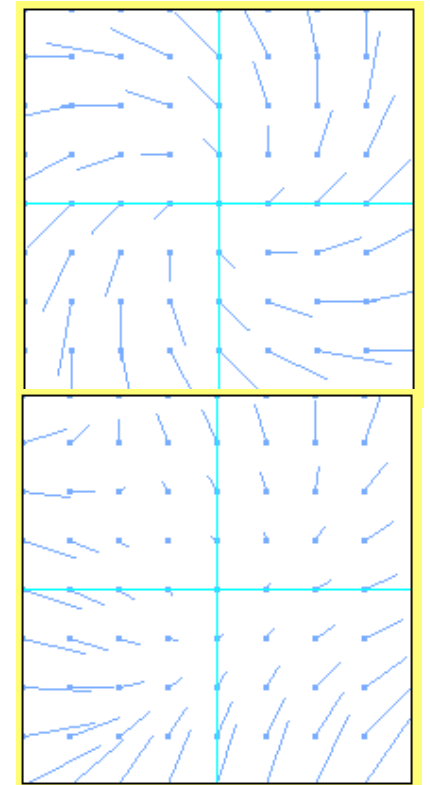


$$\nabla \cdot \mathbf{B} = 0 \quad \mathbf{B} \text{ is "solenoidal"}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}(\mathbf{x}) = 2 \text{ T/m}^2$$

$$\nabla \times \mathbf{C} \neq 0$$

$$\nabla \cdot \mathbf{C} \neq 0$$



(see <http://www.math.gatech.edu/~carlen/2507/notes/vectorCalc/dcvisualize.html>)

Del identity

■ Prove $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$

Let $\mathbf{G}(\mathbf{x}) = \nabla \times \mathbf{F}(\mathbf{x})$

Similar to
 $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$
Identity

$$\begin{aligned} [\nabla \times (\nabla \times \mathbf{F})]_i &= [\nabla \times \mathbf{G}(\mathbf{x})]_i = \epsilon_{ijk} \frac{\partial G_k}{\partial x_j} \\ &= \epsilon_{ijk} \epsilon_{klm} \frac{\partial^2 F_m}{\partial x_j \partial x_l} \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \frac{\partial^2 F_m}{\partial x_j \partial x_l} \\ &= \frac{\partial}{\partial x_i} \left(\frac{\partial F_j}{\partial x_j} \right) - \frac{\partial^2 F_i}{\partial x_j \partial x_j} \end{aligned}$$

Replace

$$G_k = \epsilon_{klm} \frac{\partial F_m}{\partial x_l}$$

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$$

A tougher Del identity

- Prove
$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A}$$

Don't start with left hand side ! Expansion will be too hard.

$$\begin{aligned} [\mathbf{A} \times (\nabla \times \mathbf{B})]_i &= \varepsilon_{ijk} A_j \left(\varepsilon_{klm} \frac{\partial B_m}{\partial x_l} \right) \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) A_j \frac{\partial B_m}{\partial x_l} \\ &= A_j \frac{\partial B_j}{\partial x_i} - A_j \frac{\partial B_i}{\partial x_j} \\ [\mathbf{B} \times (\nabla \times \mathbf{A})]_i &= B_j \frac{\partial A_j}{\partial x_i} - B_j \frac{\partial A_i}{\partial x_j} \end{aligned}$$

$$[(\mathbf{A} \cdot \nabla) \mathbf{B}]_i = A_j \frac{\partial B_i}{\partial x_j}$$

$$[(\mathbf{B} \cdot \nabla) \mathbf{A}]_i = B_j \frac{\partial A_i}{\partial x_j}$$

$$[\nabla(\mathbf{A} \cdot \mathbf{B})]_i = \frac{\partial A_j}{\partial x_i} B_j + A_j \frac{\partial B_j}{\partial x_i}$$

Not so tough (Problem 2.22)

- Given $h(\mathbf{x}) = (\mathbf{x} \times \mathbf{A}) \cdot (\mathbf{x} \times \mathbf{B})$ \mathbf{A} & \mathbf{B} are constants, $\mathbf{x} = x_i \hat{\mathbf{e}}_i$
- Prove $\nabla h(\mathbf{x}) = \mathbf{A} \times (\mathbf{x} \times \mathbf{B}) + \mathbf{B} \times (\mathbf{x} \times \mathbf{A})$

$$\begin{aligned} h(\mathbf{x}) &= (\mathbf{x} \times \mathbf{A}) \cdot (\mathbf{x} \times \mathbf{B}) = \varepsilon_{ijk} x_j A_k \varepsilon_{ilm} x_l B_m \\ &= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) x_j x_l A_k B_m = x_j^2 A_k B_k - x_j B_j x_k A_k \\ [\nabla h(\mathbf{x})]_i &= \frac{\partial}{\partial x_i} (x_j^2 A_k B_k - x_j B_j x_k A_k) \\ &= 2x_i A_k B_k - B_i x_k A_k - x_j B_j A_i \end{aligned}$$

$$\begin{aligned} \nabla h(\mathbf{x}) &= 2\mathbf{x}(\mathbf{A} \cdot \mathbf{B}) - \mathbf{B}(\mathbf{x} \cdot \mathbf{A}) - \mathbf{A}(\mathbf{x} \cdot \mathbf{B}) \\ &= [\mathbf{x}(\mathbf{A} \cdot \mathbf{B}) - \mathbf{B}(\mathbf{A} \cdot \mathbf{x})] + [\mathbf{x}(\mathbf{A} \cdot \mathbf{B}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{x})] \end{aligned}$$

$$\nabla h(\mathbf{x}) = \mathbf{A} \times (\mathbf{x} \times \mathbf{B}) + \mathbf{B} \times (\mathbf{x} \times \mathbf{A})$$

(lecture 3)

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

Potential enabling identities

- Electric **scalar potential** $f = V$ follows from

$$\nabla \times (\nabla f) = 0$$

$$[\nabla \times (\nabla f)]_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_k} \right) = 0$$

$$\epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_k} = -\epsilon_{ikj} \frac{\partial}{\partial x_k} \frac{\partial f}{\partial x_j} = -\epsilon_{ijk} \frac{\partial^2 f}{\partial x_j \partial x_k} = 0$$

Partials can be
taken in any order

- Magnetic **vector potential** $\mathbf{F} = \mathbf{A}$ follows from

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial x_i} \left(\epsilon_{ijk} \frac{\partial F_k}{\partial x_j} \right) = \epsilon_{ijk} \frac{\partial^2 F_k}{\partial x_i \partial x_j} = 0$$

ϵ_{ijk} is **antisymmetric**, and $\frac{\partial^2 F_k}{\partial x_i \partial x_j}$ symmetric

Some double
Dels $\neq 0$

$$\nabla \times (\nabla \times \mathbf{F}) \neq 0$$

$$\nabla (\nabla \cdot \mathbf{F}) \neq 0$$

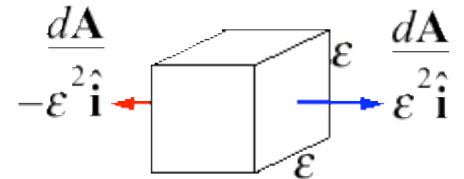
Gauss's Theorem

- Divergence definition

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^3} \oint_S \mathbf{E} \cdot d\mathbf{A} = \nabla \cdot \mathbf{E}$$

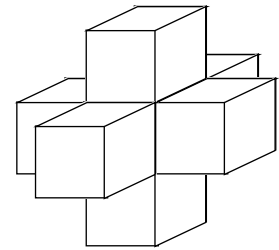
$$\mathbf{E} \cdot d\mathbf{A}$$

"flux" of \mathbf{E}
through $d\mathbf{A}$

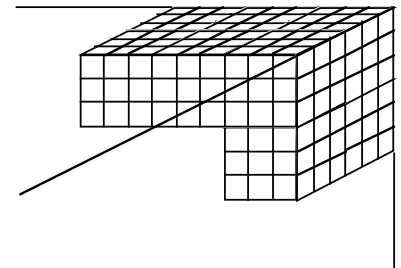


- "Proof" of Gauss's Theorem

- Place 6 tiny cubes against the original cube.
- On adjacent faces there is a common \mathbf{E} , but an opposite sign for each $d\mathbf{A}$, thus $\mathbf{E} \cdot d\mathbf{A}$ cancels.
- Summing over a block broken into tiny cubes, the surface fluxes cancel except on the block's surface.



$$\sum_{i=1}^N \left[\oint_{S_i} \mathbf{E} \cdot d\mathbf{A} \right] = \sum_{i=1}^N \left[\nabla \cdot \mathbf{E}(\mathbf{x}_i) \right] \epsilon^3$$



- Gauss's Theorem for a closed surface S

Only the external
surface flux left

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \oint_V \nabla \cdot \mathbf{E}(\mathbf{x}) d^3x$$

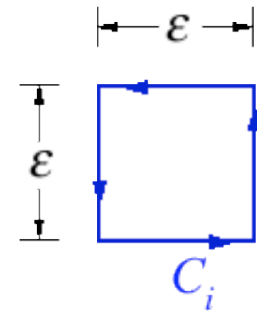
Divergence integrated
over the volume inside S

Stokes's theorem

■ Curl definition

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} \oint_C \mathbf{B} \cdot d\ell = [\nabla \times \mathbf{B}] \cdot \hat{\mathbf{n}}$$

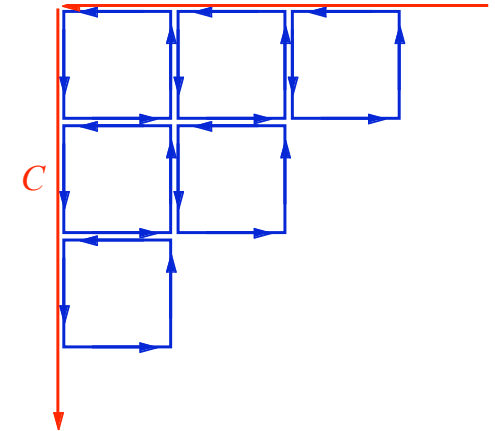
$\mathbf{B} \cdot d\ell$ "line element of \mathbf{B} along $d\ell$ "
 $\hat{\mathbf{n}}$ "normal to surface bounded by curve"



■ "Proof" of Stokes's Theorem

- Break large loop C into many tiny loops.
- On adjacent sides there is a common \mathbf{B} but an opposite sign for each $d\ell$ thus line elements cancel.
- The tiny loop line elements cancel except those on the large loop.

$$\sum_{i=1}^N \left[\oint_{C_i} \mathbf{B} \cdot d\ell \right] = \sum_{i=1}^N [\nabla \times \mathbf{B}(\mathbf{x}_i)] \cdot \hat{\mathbf{n}} \epsilon^2$$



■ Stokes's Theorem for a closed curve C

Only line elements on C remain

$$\oint_C \mathbf{B} \cdot d\ell = \oint_S [\nabla \times \mathbf{B}(\mathbf{x})] \cdot d\mathbf{A}$$

Curl integrated over any surface S bounded by C

Summary of integral theorems

■ Gauss's Theorem

Flux integrated over
closed surface S = Divergence integrated
over the volume inside S

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \oint_V \nabla \cdot \mathbf{E}(\mathbf{x}) d^3x$$

Neither side looks particularly simple, however, for static electric fields the right hand side will be shown to be equal to the enclosed charge/ ϵ_0

■ Stokes's Theorem

Line elements integrated
over closed curve = Curl integrated over any
surface S bounded by C

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \oint_S [\nabla \times \mathbf{B}(\mathbf{x})] \cdot d\mathbf{A}$$

RHR on C gives
normal to surface S

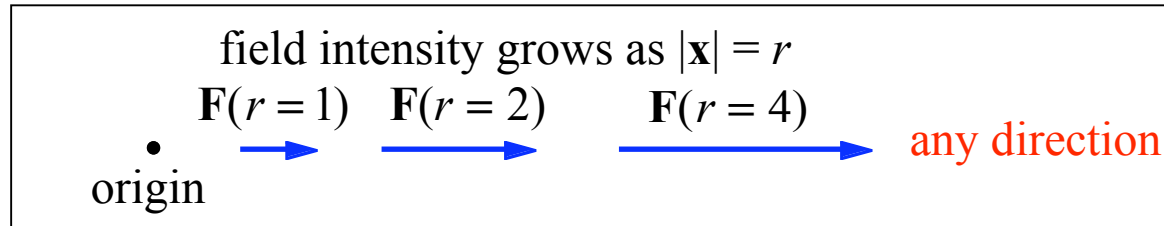
Neither side looks particularly simple, however, for constant magnetic fields the right hand side will be shown to be equal to the μ_0 •current enclosed.

Problem 2.10

- Consider the vector field

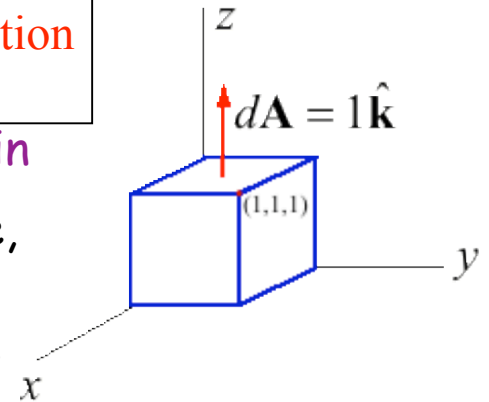
$$\mathbf{F}(\mathbf{x}) = \mathbf{x} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

- 1) Sketch the field



- 2) Flux through cube, edge length = 1, corner at the origin

- Field in x-z, y-z, or x-y plane is **parallel** to a cube face, therefore, through those 3 faces the flux = 0
- The same flux goes through each of the other 3 faces
- Consider the top face: $z = 1$, area = 1.



$$\oint_{Top} \mathbf{F} \cdot d\mathbf{A} = \mathbf{x} \cdot \hat{\mathbf{k}} = z = 1$$

$$\oint_S \mathbf{F} \cdot d\mathbf{A} = 3$$

- 3) Using Gauss's theorem

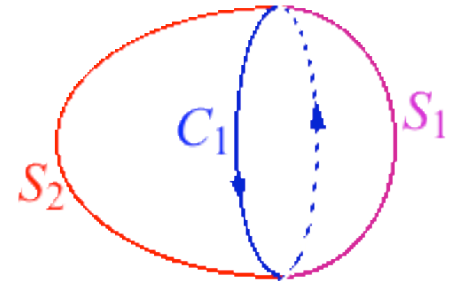
$$\nabla \cdot \mathbf{F} = \frac{\partial F_i}{\partial x_i} = 1 + 1 + 1 = 3$$

$$\oint_S \mathbf{F} \cdot d\mathbf{A} = \oint_V \nabla \cdot \mathbf{F} dV = 3 \oint_V dV = 3$$

Problem 2.11

- Use Gauss's then Stokes's Theorem to prove:

$$\oint_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = -\oint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$$



Gauss's Theorem Let $S = S_1 + S_2$, and $\mathbf{G} = \nabla \times \mathbf{F}$

$$\oint_S \mathbf{G} \cdot d\mathbf{A} = \oint_V (\nabla \cdot \mathbf{G}) dV$$

See Slide 6 this lecture

$$\oint_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{A} + \oint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = \oint_V \nabla \cdot (\nabla \times \mathbf{F}) dV = 0$$

Stokes's Theorem

$$\oint_{S_1} (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} dA = \oint_{C_1} \mathbf{F} \cdot d\boldsymbol{\ell}$$

$$\oint_{S_2} (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} dA = \oint_{C_2} \mathbf{F} \cdot d\boldsymbol{\ell}$$

$$\oint_{C_1} = -\oint_{C_2}$$

S_1 outward normal via right hand rule on C_1 .
 S_2 outward normal opposite direction for C_2

Gauss's & Stokes's Theorems give the same answer

$$\oint_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = -\oint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$$

Problem 2.9

- Consider the vector field

$$\mathbf{F}(\mathbf{x}) = \hat{\mathbf{k}} \times \mathbf{x}$$

$$\mathbf{F}(\mathbf{x}) = \hat{\mathbf{k}} \times (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}} = r\hat{\phi}$$

Phi direction with
no z dependence

b) Line integral over circle radius a

$$\oint_C \mathbf{F} \cdot d\ell = \int_0^{2\pi} (a\hat{\phi}) \cdot (a d\phi \hat{\phi}) = \underline{2\pi a^2}$$

c) Line integral via Stokes's Theorem

$$[\nabla \times \mathbf{F}]_3 = \varepsilon_{3jk} \frac{\partial F_k}{\partial x_j} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 2$$

$$\nabla \times \mathbf{F} = 2\hat{\mathbf{k}}$$

Stokes's Theorem

$$\oint_C \mathbf{F} \cdot d\ell = \oint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = 2\hat{\mathbf{k}} \cdot \hat{\mathbf{k}} \int_0^{2\pi} d\phi \int_0^a r dr = \underline{2\pi a^2}$$

a) Sketch the field

