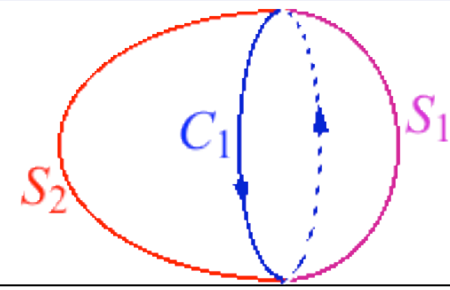
PHY481: Electromagnetism

Vector operators in curvilinear coordinates

Vector field Problem 2.11

- Use Gauss's then Stokes's Theorem to prove:

$$\oint_{S_2} (\nabla\times\mathbf{F})\cdot d\mathbf{A} = -\oint_{S_1} (\nabla\times\mathbf{F})\cdot d\mathbf{A}$$



Gauss's Theorem Let $S = S_1 + S_2$, and $\mathbf{G} = \nabla\times\mathbf{F}$

$$\oint_S \mathbf{G}\cdot d\mathbf{A} = \oint_V (\nabla\cdot\mathbf{G})\,dV$$

See Slide 6 previous lecture

$$\oint_{S_2} (\nabla\times\mathbf{F})\cdot d\mathbf{A} + \oint_{S_1} (\nabla\times\mathbf{F})\cdot d\mathbf{A} = \oint_V \nabla\cdot(\nabla\times\mathbf{F})\,dV = 0$$

Stokes's Theorem

$$\oint_{S_1} (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dA = \oint_{C_1} \mathbf{F}\cdot d\ell$$

$$\oint_{S_2} (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dA = \oint_{C_2} \mathbf{F}\cdot d\ell$$

$$\oint_{C_1} = -\oint_{C_2}$$

S_1 outward normal via right hand rule on C_1. S_2 outward normal opposite direction for C_2

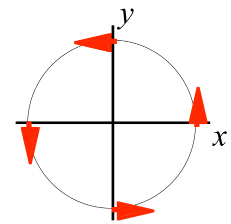
Gauss's & Stokes's Theorems give the same answer

$$\oint_{S_2} (\nabla\times\mathbf{F})\cdot d\mathbf{A} = -\oint_{S_1} (\nabla\times\mathbf{F})\cdot d\mathbf{A}$$

Vector field Problem 2.9

- Consider the vector field $\mathbf{F}(\mathbf{x}) = \hat{\mathbf{k}} \times \mathbf{x}$

$$\mathbf{F}(\mathbf{x}) = \hat{\mathbf{k}} \times \left(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} \right) = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} = r\hat{\boldsymbol{\phi}}$$

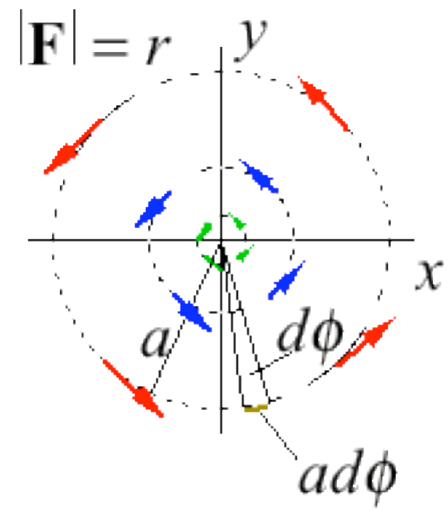


ϕ direction
no z-dependence

b) Line integral over circle radius a

$$\oint_C \mathbf{F} \cdot d\ell = \int_0^{2\pi} \left(a\hat{\boldsymbol{\phi}} \right) \cdot \left(a\, d\phi\, \hat{\boldsymbol{\phi}} \right) = \underline{2\pi a^2}$$

a) Sketch the field



c) Line integral via Stokes's Theorem

$$[\nabla \times \mathbf{F}]_3 = \varepsilon_{3jk} \frac{\partial F_k}{\partial x_j} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 2$$

$$\nabla \times \mathbf{F} = 2\hat{\mathbf{k}}$$

Stokes's Theorem

$$\oint_C \mathbf{F} \cdot d\ell = \oint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A} \qquad = 2\hat{\mathbf{k}} \cdot \hat{\mathbf{k}} \int_0^{2\pi} d\phi \int_0^a r\,dr = \underline{2\pi a^2}$$

Curvilinear coordinates

- Cartesian coordinate vector operators

Del	Gradient	Divergence	Curl	Laplacian
$\nabla = \hat{\mathbf{e}}_i \dfrac{\partial}{\partial x_i}$	$\nabla V(\mathbf{x}) = \dfrac{\partial V}{\partial x_i}\hat{\mathbf{e}}_i$	$\nabla \cdot \mathbf{E}(\mathbf{x}) = \dfrac{\partial E_i}{\partial x_i}$	$[\nabla \times \mathbf{B}(\mathbf{x})]_i = \varepsilon_{ijk}\dfrac{\partial B_k}{\partial x_j}$	$\nabla^2 V = \dfrac{\partial^2 V}{\partial x_i^2}$

- Compare coordinate systems - SCALE FACTORS

Coordinates u_1, u_2, u_3	Unit vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$	Position $\mathbf{x} =$	Displacement $d\mathbf{s} =$	Scale Factors h_1, h_2, h_3
Cartesian x, y, z	$\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$	$x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$	$dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$	$1, 1, 1$
Cylindrical r, ϕ, z	$\hat{\mathbf{r}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{k}}$	$r\,\hat{\mathbf{r}} + z\,\hat{\mathbf{k}}$	$dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{k}}$	$1, r, 1$
Spherical r, θ, ϕ	$\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$	$r\,\hat{\mathbf{r}}$	$dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$	$1, r, r\sin\theta$

Scale factors

- Scale factors relate displacements to coordinate changes

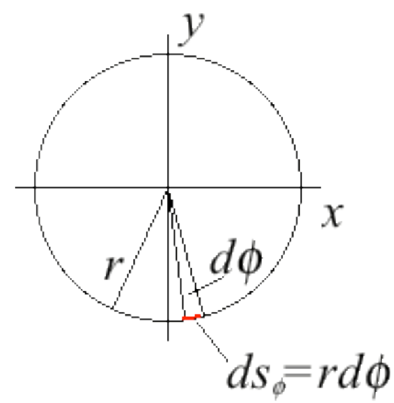
$$d\mathbf{s} = h_1 du_1\, \hat{\mathbf{e}}_1 + h_2 du_2\, \hat{\mathbf{e}}_2 + h_3 du_3\, \hat{\mathbf{e}}_3$$

$$h_i = \frac{ds_i}{du_i}$$

- Example: cylindrical coordinate scale factor

$$(u_1, u_2, u_3) = (r, \phi, z)$$

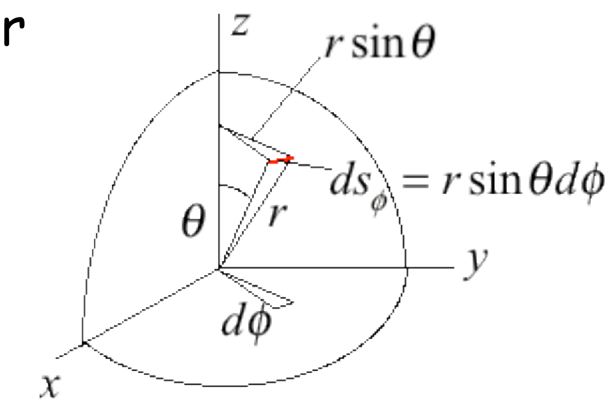
$$ds_\phi = r d\phi; \qquad h_\phi = \frac{ds_\phi}{d\phi} = r$$



- Example: spherical coordinate scale factor

$$(u_1, u_2, u_3) = (r, \theta, \phi)$$

$$ds_\phi = r\sin\theta d\phi; \qquad h_\phi = \frac{ds_\phi}{d\phi} = r\sin\theta$$



Vector operators in curvilinear coordinates

- Operators in terms of coordinates and scale factors

$$\nabla V(\mathbf{x}) = \frac{1}{h_i}\frac{\partial V}{\partial u_i}\hat{\mathbf{e}}_i$$

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 E_1) + \frac{\partial}{\partial u_2}(h_1 h_3 E_2) + \frac{\partial}{\partial u_3}(h_1 h_2 E_3)\right]$$

$$\nabla^2 V(\mathbf{x}) = \frac{1}{h_1 h_2 h_3}\sum_{(ijk)}\frac{\partial}{\partial u_i}\left[\frac{h_j h_k}{h_i}\frac{\partial V}{\partial u_i}\right]$$

(ijk) = cyclic permutations of (123)

$$\left[\nabla \times \mathbf{B}(\mathbf{x})\right]_i = \varepsilon_{ijk}\left[\frac{1}{h_j h_k}\frac{\partial}{\partial x_j}\left(h_k B_k\right)\right]$$

Expanding curl in curvilinear coordinates

Cylindrical

$\hat{\mathbf{r}}$ $$[\nabla\times\mathbf{B}(\mathbf{x})]_r = \left[\frac{1}{r}\frac{\partial}{\partial\phi}(B_z) - \frac{\partial}{\partial z}(B_\phi)\right]$$

$\hat{\boldsymbol{\phi}}$ $$[\nabla\times\mathbf{B}(\mathbf{x})]_\phi = \left[\frac{\partial}{\partial z}(B_r) - \frac{\partial}{\partial r}(B_z)\right]$$

$\hat{\mathbf{k}}$ $$[\nabla\times\mathbf{B}(\mathbf{x})]_z = \frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\phi) - \frac{\partial}{\partial\phi}(B_r)\right]$$

Spherical

$\hat{\mathbf{r}}$ $$[\nabla\times\mathbf{B}(\mathbf{x})]_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta B_\phi) - \frac{\partial}{\partial\phi}(B_\theta)\right]$$

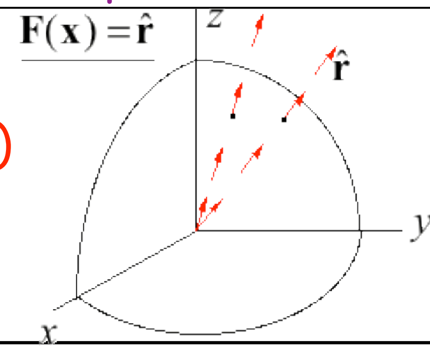
$\hat{\boldsymbol{\theta}}$ $$[\nabla\times\mathbf{B}(\mathbf{x})]_\theta = \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(B_r) - \frac{\partial}{\partial r}(rB_\phi)\right]$$

$\hat{\boldsymbol{\phi}}$ $$[\nabla\times\mathbf{B}(\mathbf{x})]_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial}{\partial\theta}(B_r)\right]$$

Example 2.7

- In spherical coordinates, calculate divergence and curl & interpret

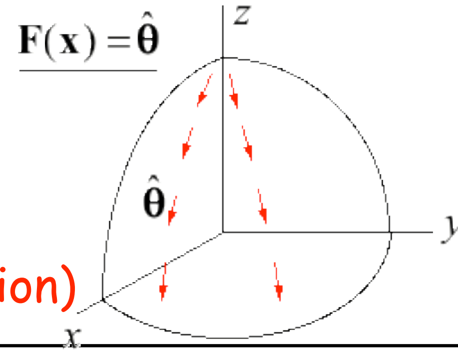
$\mathbf{F}(\mathbf{x}) = 1\hat{\mathbf{r}}$ (constant F in radial direction)



$$\nabla \cdot \hat{\mathbf{r}} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{dr}\left(r^2 \sin\theta\, F_r\right)\right] = \frac{2}{r}$$ (field lines spread in r)

$$\nabla \times \hat{\mathbf{r}} = 0 \quad (\text{as } F_\theta = F_\phi = 0)$$ (field has no “vorticity”)

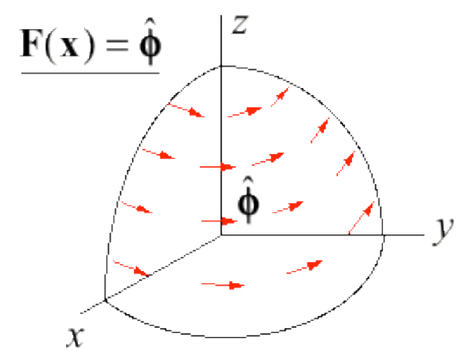
$\mathbf{F}(\mathbf{x}) = 1\hat{\boldsymbol{\theta}}$ (constant F in polar direction)



$$\nabla \cdot \hat{\boldsymbol{\theta}} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{d\theta}\left(r\sin\theta F_\theta\right)\right] = \frac{\cot\theta}{r}$$ (field lines spread poles to equator)

$$\nabla \times \hat{\boldsymbol{\theta}} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_\theta\right)\right]\hat{\boldsymbol{\phi}} = \frac{1}{r}\hat{\boldsymbol{\phi}}$$ (field curls around ϕ direction)

$\mathbf{F}(\mathbf{x}) = 1\hat{\boldsymbol{\phi}}$ (constant F in azimuth direction)



$$\nabla \cdot \hat{\boldsymbol{\phi}} = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{d\phi}\left(rF_\phi\right)\right] = 0$$ (field lines are parallel)

$$\nabla \times \hat{\boldsymbol{\phi}} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta F_\phi\right)\right]\hat{\mathbf{r}} + \frac{1}{r}\left[-\frac{\partial}{\partial r}\left(rF_\phi\right)\right]\hat{\boldsymbol{\theta}} = \frac{1}{r\sin\theta}\hat{\mathbf{k}}$$

$\hat{\mathbf{k}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$ (see problem 2.14) (field curls around z direction)

Helmholtz Theorem

- Important identities $\nabla \times (\nabla \psi) = 0$ $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
- Arbitrary function: $\mathbf{H}(\mathbf{x}) = \mathbf{F}(\mathbf{x}) + \mathbf{G}(\mathbf{x})$

"irrotational" part $\mathbf{F}(\mathbf{x}) = -\nabla \psi(\mathbf{x})$

"solenoidal" part $\mathbf{G}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x})$

Helmholtz Theorem

Let $\nabla \times \mathbf{H}(\mathbf{x}) = \mathbf{c}(\mathbf{x})$, $\nabla \cdot \mathbf{H}(\mathbf{x}) = d(\mathbf{x})$

and $\mathbf{c}(\mathbf{x})$ & $d(\mathbf{x}) \underset{r\to\infty}{\longrightarrow} 0$ faster than r^{-2} and $\mathbf{H}(\mathbf{x}) \underset{r\to\infty}{\longrightarrow} 0$

$$\mathbf{H}(\mathbf{x}) = -\nabla \psi + \nabla \times \mathbf{A}$$

$$\psi(\mathbf{x}) = \int \frac{d(\mathbf{x}')d^3x'}{4\pi|\mathbf{x}-\mathbf{x}'|}$$

$$\mathbf{A}(\mathbf{x}) = \int \frac{\mathbf{c}(\mathbf{x}')d^3x'}{4\pi|\mathbf{x}-\mathbf{x}'|}$$

$d(\mathbf{x})$ will be charge density/ε_0

$\mathbf{c}(\mathbf{x})$ will be μ_0 • current density