PHY481: Electromagnetism

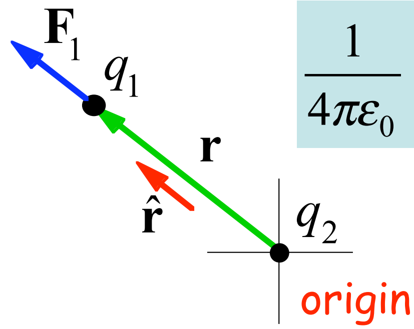
Coulomb's law and the electric field

Coulomb's Law

- Force between two charges

Lectures 1 - 2

$$\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}$$



$$\frac{1}{4\pi\varepsilon_0} = 9.0\times 10^9 \ \frac{\text{Nm}^2}{\text{C}^2}$$

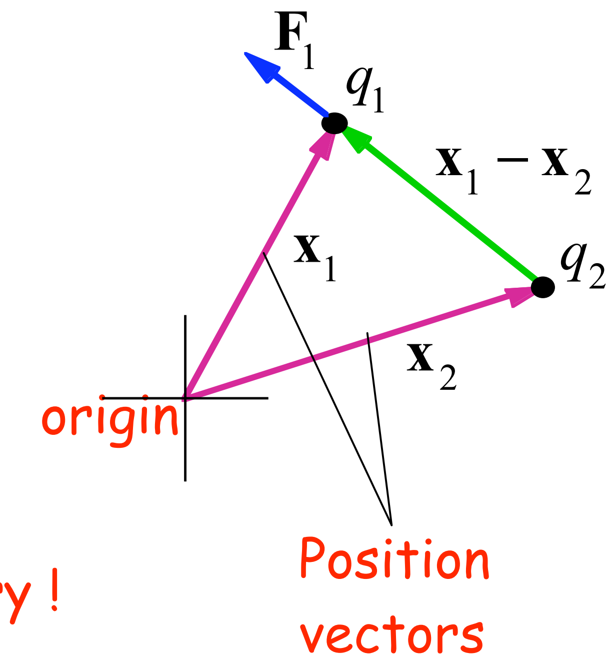
Not a typo !

From now on,

$$\mathbf{F}_1 = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\mathbf{x}_1 - \mathbf{x}_2}{\left|\mathbf{x}_1 - \mathbf{x}_2\right|^3}$$

Unit vector with a direction from 2 to 1

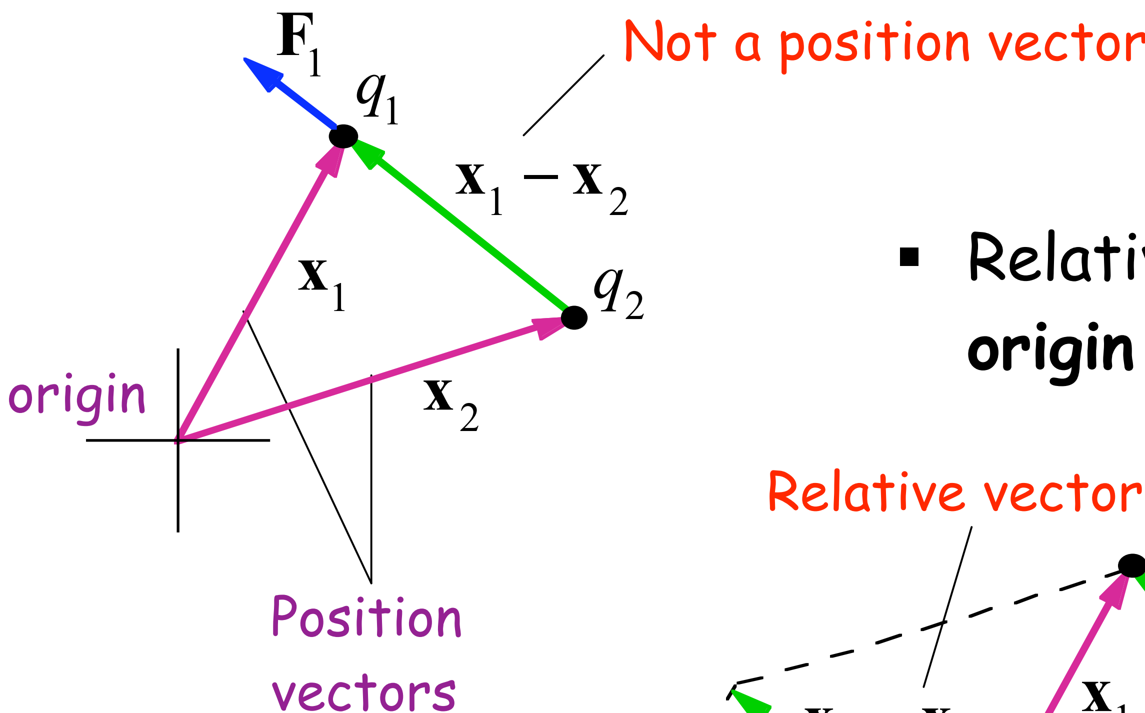
$$\frac{\mathbf{x}_1 - \mathbf{x}_2}{\left|\mathbf{x}_1 - \mathbf{x}_2\right|}$$



Erase previous version from your memory !

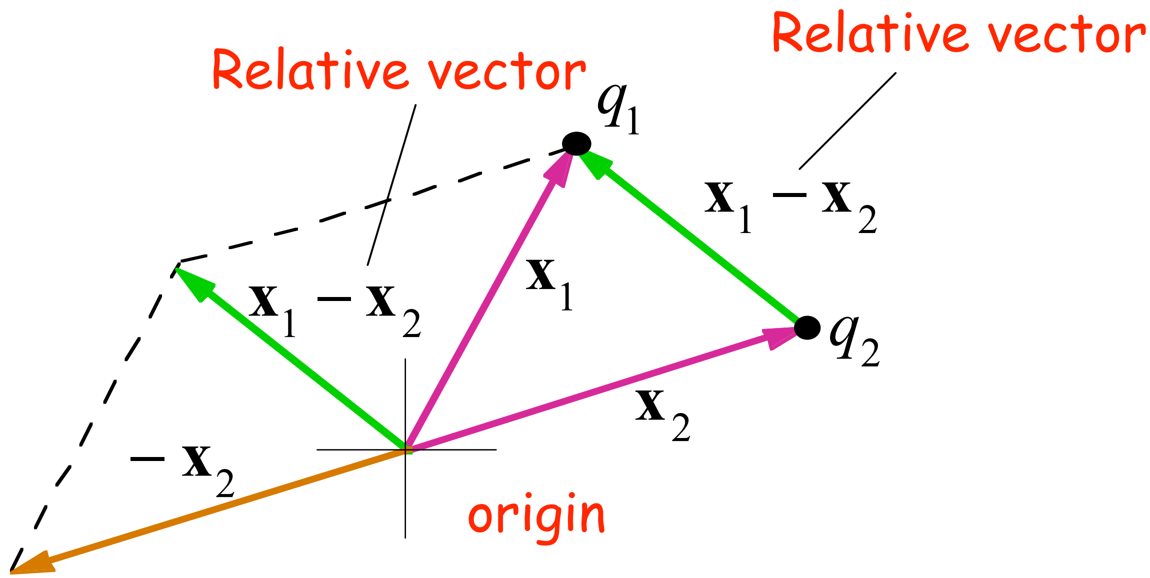
Why change notation?

- Position vectors w.r.t. a fixed origin.



- Relative vectors are **origin independent**

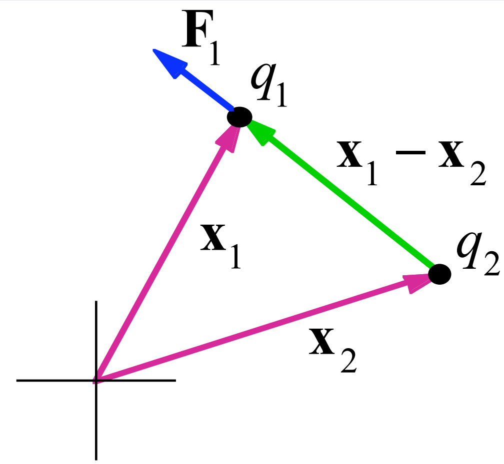
Construct $\mathbf{x}_1$-$\mathbf{x}_2$



Multiple charges

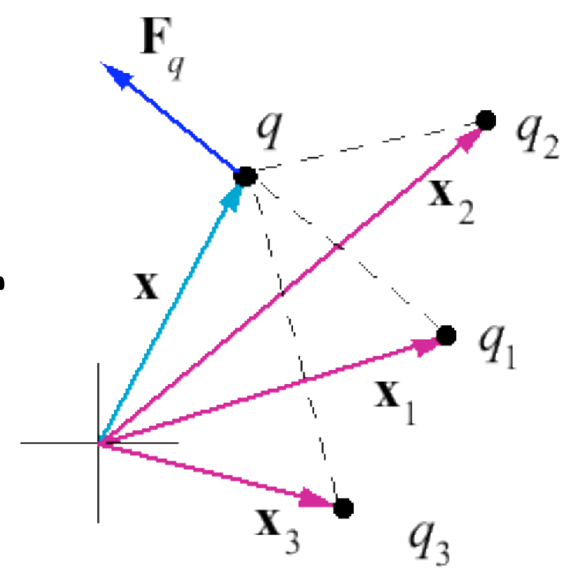
- Coulomb's law (2 charges)

$$\mathbf{F}_1 = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3}$$



- Force on charge q from N charges, $q_1 \cdots q_N$

$$\mathbf{F}_q(\mathbf{x}) = \sum_{k=1}^{N} \frac{q q_k}{4\pi\varepsilon_0} \frac{\mathbf{x} - \mathbf{x}_k}{|\mathbf{x} - \mathbf{x}_k|^3}$$



- Reserve **r** for the position vector in spherical coordinates

$$\mathbf{x} = \mathbf{r} = r\,\hat{\mathbf{r}}$$

Electric field

- Coulomb's law for discrete charges q, and q_k , k=1,...N

$$\mathbf{F}_q(\mathbf{x}) = \sum_{k=1}^{N} \frac{q q_k}{4\pi\varepsilon_0} \frac{\mathbf{x} - \mathbf{x}_k}{|\mathbf{x} - \mathbf{x}_k|^3}$$

- Use a very small charge *q* at the point **x** to measure the force vector $\mathbf{F}_q$. The electric field **E** at the point **x** is the limit

$$\mathbf{E}(\mathbf{x}) = \lim_{q \to 0} \frac{\mathbf{F}_q}{q}$$

- For discrete charges q, and q_k , k=1,...N the field is

$$\mathbf{E}(\mathbf{x}) = \sum_{k=1}^{N} \frac{q_k}{4\pi\varepsilon_0} \frac{\mathbf{x} - \mathbf{x}_k}{|\mathbf{x} - \mathbf{x}_k|^3}$$

- At the point **x**, an electric field **E**(**x**) exerts a force **F** on a charge *q* such that

$$\mathbf{F}_q = q\mathbf{E}(\mathbf{x})$$

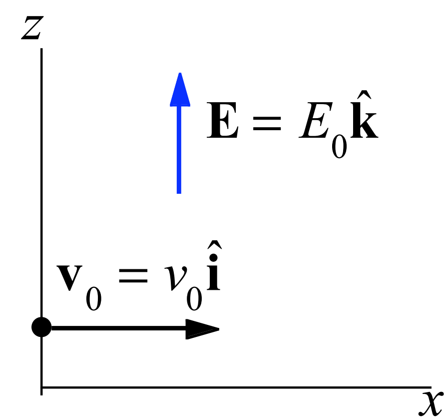
Charge motion in an electric field

In an electric field E_0 in the z direction, a proton travels with an initial velocity v_0 in the x direction. What is its trajectory?

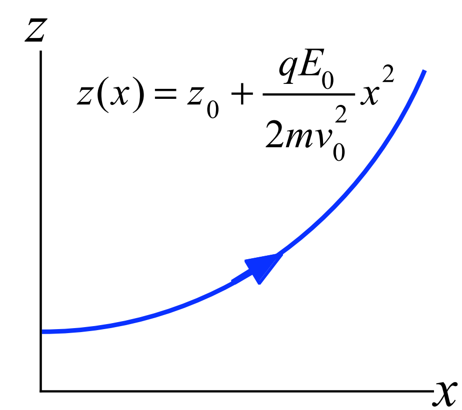
Acceleration: $$\mathbf{a} = \frac{\mathbf{F}}{m} = \frac{q\mathbf{E}}{m} = \frac{qE_0}{m}\hat{\mathbf{k}}$$

Displacement: $$\mathbf{s}(t) = \mathbf{v}_0 t + \tfrac{1}{2}\mathbf{a}t^2$$

$$= (v_0 t)\hat{\mathbf{i}} + \frac{qE_0 t^2}{2m}\hat{\mathbf{k}}$$



Position: $$x = x_0 + v_0 t; \quad z = z_0 + \frac{qE_0}{2m}t^2$$



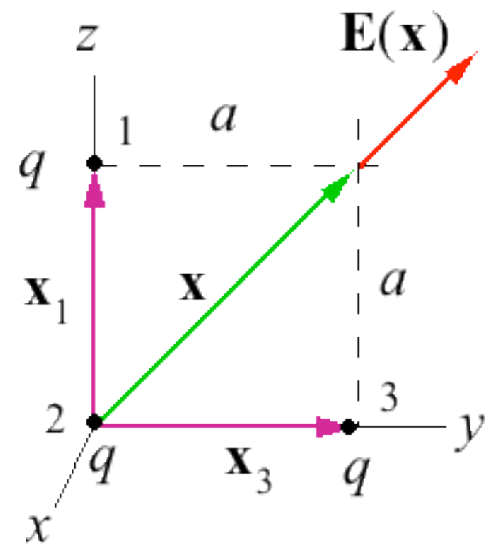
Trajectory is coupled:

$$z(x) = z_0 + \frac{qE_0}{2mv_0^2}(x - x_0)^2 \quad \text{(Parabola)}$$

Electric field calculation

Charge q at 3 corners of square side a.

Find Electric field at 4th corner



Position vectors

$$\mathbf{x} = a(\hat{\mathbf{j}} + \hat{\mathbf{k}}) \quad \mathbf{x}_1 = a\hat{\mathbf{k}}, \mathbf{x}_2 = 0, \mathbf{x}_3 = a\hat{\mathbf{j}}$$

Relative vectors

$$\mathbf{x} - \mathbf{x}_1 = a(\hat{\mathbf{j}} + \hat{\mathbf{k}}) - a\hat{\mathbf{k}} = a\hat{\mathbf{j}}$$

$$\mathbf{x} - \mathbf{x}_2 = a(\hat{\mathbf{j}} + \hat{\mathbf{k}}) - 0 = a(\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

$$\mathbf{x} - \mathbf{x}_3 = a(\hat{\mathbf{j}} + \hat{\mathbf{k}}) - a\hat{\mathbf{j}} = a\hat{\mathbf{k}}$$

Electric field

$$\mathbf{E}(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0} \sum_{k=1}^{N} \frac{\mathbf{x} - \mathbf{x}_k}{|\mathbf{x} - \mathbf{x}_k|^3}$$

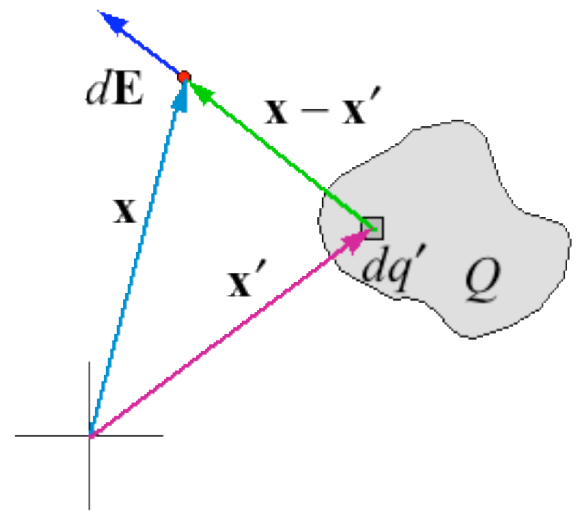
$$= \frac{q}{4\pi\varepsilon_0} \left[\frac{a\hat{\mathbf{j}}}{a^3} + \frac{a(\hat{\mathbf{j}} + \hat{\mathbf{k}})}{2\sqrt{2}a^3} + \frac{a\hat{\mathbf{k}}}{a^3} \right] = \underline{\frac{q}{4\pi\varepsilon_0 a^2} \left(1 + \frac{\sqrt{2}}{4} \right) (\hat{\mathbf{j}} + \hat{\mathbf{k}})}$$

Continuous charge distributions

Distribution:	Volume	Surface	Line
Charge density:	$\rho(\mathbf{x}')$	$\sigma(\mathbf{x}')$	$\lambda(\mathbf{x}')$
Total Charge:	$Q = \int \rho(\mathbf{x}') d^3x'$	$Q = \int \sigma(\mathbf{x}') d^2x'$	$Q = \int \lambda(\mathbf{x}') dx'$

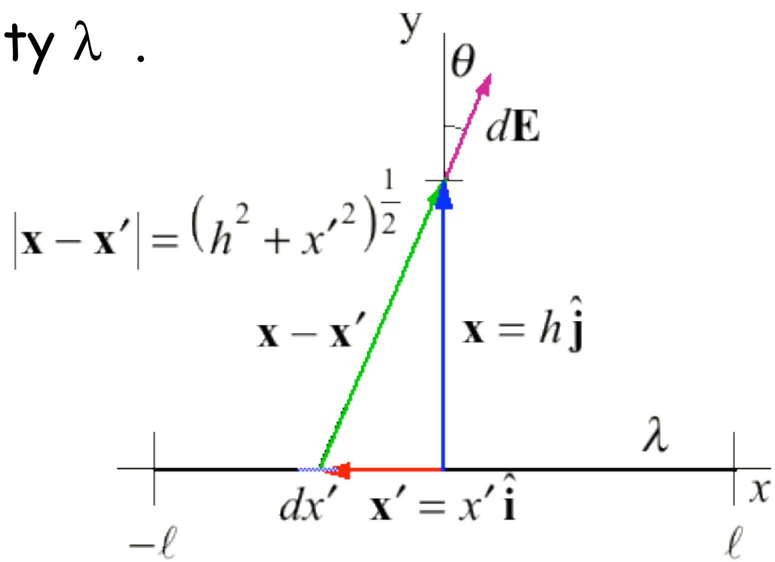
- Integration over charge distribution

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \rho(\mathbf{x}') d^3x'$$



Linear charge distributions

What is the electric field at a distance h above the bisector of a line length 2*l*, with charge density λ .



Direct substitution

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x'$$

$$\mathbf{E}(\mathbf{x}) = \frac{\lambda}{4\pi\varepsilon_0}\int (h\hat{\mathbf{j}} - x'\,\hat{\mathbf{i}})\left(h^2 + x'^2\right)^{-\frac{3}{2}} dx'$$

Odd Integrand

$$= \frac{\lambda h\hat{\mathbf{j}}}{4\pi\varepsilon_0}\left[\int_{-\ell}^{\ell}\left(h^2 + x'^2\right)^{-\frac{3}{2}} dx'\right] - \frac{\lambda\hat{\mathbf{i}}}{4\pi\varepsilon_0}\left[\int_{-\ell}^{\ell} x'\left(h^2 + x'^2\right)^{-\frac{3}{2}} dx'\right]$$

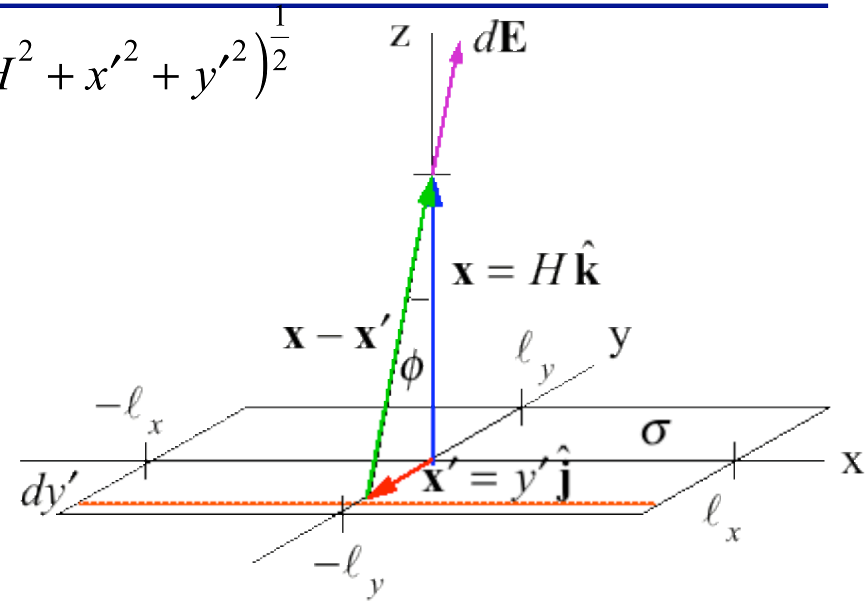
$$E_y(h) = \frac{\lambda}{4\pi\varepsilon_0 h}\left[x'\left(h^2 + x'^2\right)^{-\frac{1}{2}}\right]_{-\ell}^{+\ell} = \frac{\lambda\ell}{2\pi\varepsilon_0 h\left(h^2 + \ell^2\right)^{1/2}}$$

Surface charge distribution

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x' \qquad |\mathbf{x}-\mathbf{x}'| = \left(H^2 + x'^2 + y'^2\right)^{\frac{1}{2}}$$

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{H\hat{\mathbf{k}} - y'\hat{\mathbf{j}} - x'\hat{\mathbf{i}}}{\left(H^2 + x'^2 + y'^2\right)^{\frac{3}{2}}}\sigma dx'dy'$$

$-y'\hat{\mathbf{j}} - x'\hat{\mathbf{i}}$ terms are odd



$$E_z = \frac{\sigma H}{4\pi\varepsilon_0}\int_{-\ell}^{\ell} dy' \int_{-\ell}^{\ell} dx'\left(\left[H^2 + y'^2\right] + x'^2\right)^{-\frac{3}{2}}$$

$$= \frac{\sigma H \ell_x}{2\pi\varepsilon_0}\int_{-\ell_y}^{\ell_y} dy'\left[\frac{1}{\left[H^2 + y'^2\right]\left(\left[H^2 + \ell_x^2\right] + y'^2\right)^{\frac{1}{2}}}\right]$$

$$E_z = \frac{\sigma}{\pi\varepsilon_0}\arctan\left[\frac{\ell_x \ell_y}{H\left(H^2 + \ell_x^2 + \ell_y^2\right)^{\frac{1}{2}}}\right]$$