PHY481: Electromagnetism

HW2 and Discussion

Problem 2.2

Parllelogram Area

$$|\mathbf{A}\times\mathbf{B}| = |A_y||B_x| = \underline{|\mathbf{A}||\mathbf{B}|\sin\theta}$$

Parllelepided Volume

$$\mathbf{A}\times\mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin\theta\,\hat{\mathbf{k}}$$

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} &= (|\mathbf{A}||\mathbf{B}|\sin\theta)\hat{\mathbf{k}}\cdot\mathbf{C}\\ &= |\mathbf{A}||\mathbf{B}||\mathbf{C}|\sin\theta\cos\phi\\ &= \mathit{Volume}\end{aligned}$$

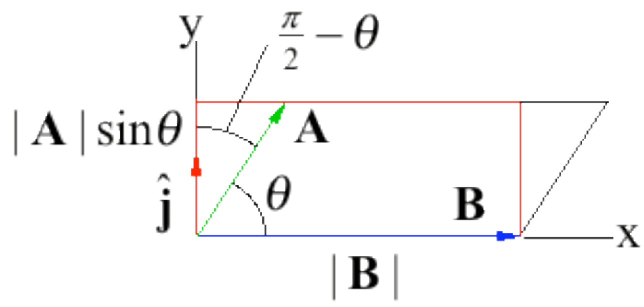
Show A•(BXC) = Volume

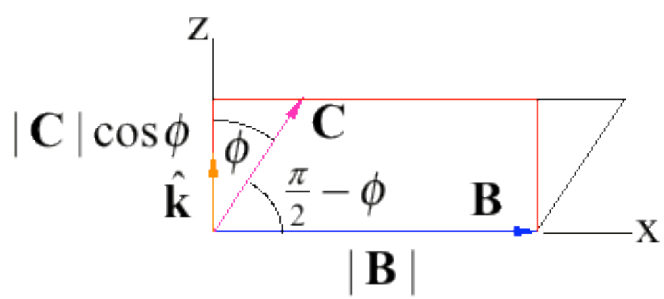
$$\begin{aligned}\mathbf{B}\times\mathbf{C} &= \hat{\mathbf{j}}|\mathbf{B}||\mathbf{C}|\sin\left(\tfrac{\pi}{2}-\phi\right)\\ &= \hat{\mathbf{j}}|\mathbf{B}||\mathbf{C}|\cos\phi\end{aligned}$$

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \mathbf{A}\cdot\hat{\mathbf{j}}|\mathbf{B}||\mathbf{C}|\cos\phi\\ &= |\mathbf{A}||\mathbf{B}||\mathbf{C}|\cos\phi\cos\left(\tfrac{\pi}{2}-\theta\right)\\ &= |\mathbf{A}||\mathbf{B}||\mathbf{C}|\cos\phi\sin\theta\\ &= \mathit{Volume}\end{aligned}$$

<u>Area of rectangle</u>

$$\mathit{Area} = |\mathbf{A}||\mathbf{B}|\sin\theta$$





<u>Volume of rectangular solid</u>

$$\mathit{Volume} = |\mathbf{A}|\sin\theta|\mathbf{B}||\mathbf{C}|\cos\phi$$

Problem 2.3

Tensor product: $$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Cross product components i & l: $$(\mathbf{A}\times\mathbf{B})_i = \varepsilon_{ijk}A_jB_k; \quad (\mathbf{C}\times\mathbf{D})_l = \varepsilon_{lmn}C_mD_n$$

Dot product: $$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = \left(\varepsilon_{ijk}A_jB_k\right)\left(\varepsilon_{imn}C_mD_n\right)$$
$$= \varepsilon_{jki}\varepsilon_{imn}A_jB_kC_mD_n$$

Replace tensor product: $$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = \left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\right)A_jB_kC_mD_n$$

Replace l,m,n with j or k: $$= A_jB_kC_jD_k - A_jB_kC_kD_j$$
$$= \left(A_jC_j\right)\left(B_kD_k\right) - \left(A_jD_j\right)\left(B_kC_k\right)$$

Identify dot products: $$\underline{(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D})} = \underline{(\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})}$$

Problem 2.6

$$\underline{\nabla\cdot(g\mathbf{F})} = \frac{\partial}{\partial x_i}(gF_i) = F_i\frac{\partial g}{\partial x_i} + g\frac{\partial F_i}{\partial x_i} = \underline{\mathbf{F}\cdot\nabla g + g\nabla\cdot\mathbf{F}}$$ Scalar!

$$[\nabla\times(g\mathbf{F})]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(gF_k) = \varepsilon_{ijk}\frac{\partial g}{\partial x_j}F_k + g\varepsilon_{ijk}\frac{\partial F_k}{\partial x_j}$$

$$\underline{\nabla\times(g\mathbf{F})} = \underline{(\nabla g)\times\mathbf{F} + g\nabla\times\mathbf{F}}$$

$$\nabla\cdot(\mathbf{F}\times\mathbf{G}) = \frac{\partial}{\partial x_i}(\varepsilon_{ijk}F_jG_k) = \varepsilon_{ijk}\left(\frac{\partial F_j}{\partial x_i}\right)G_k + \varepsilon_{ijk}F_j\frac{\partial G_k}{\partial x_i}$$

<u>Even vs Odd permutations</u>

$$\varepsilon_{ijk} = \varepsilon_{kij},\ \text{and}\ \varepsilon_{ijk} = -\varepsilon_{jik}$$

$$\underline{\nabla\cdot(\mathbf{F}\times\mathbf{G})} = G_k\varepsilon_{kij}\left(\frac{\partial F_j}{\partial x_i}\right) - F_j\varepsilon_{jik}\frac{\partial G_k}{\partial x_i} = \underline{\mathbf{G}\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times\mathbf{G})}$$

<u>Like AxBxC=B(A•C)-C(A•B)</u>

$$[\nabla\times(\nabla\times\mathbf{F})]_i = [\nabla\times\mathbf{G}]_i = \varepsilon_{ijk}\frac{\partial G_k}{\partial x_j}$$

$$G_k = (\nabla\times\mathbf{F})_k = \varepsilon_{klm}\frac{\partial F_m}{\partial x_l}$$

$$[\nabla\times(\nabla\times\mathbf{F})]_i = \varepsilon_{ijk}\varepsilon_{klm}\frac{\partial^2 F_m}{\partial x_j\partial x_l} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial^2 F_m}{\partial x_j\partial x_l}$$

$$\nabla\times(\nabla\times\mathbf{F}) = \hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}\left(\frac{\partial F_j}{\partial x_j}\right) - \frac{\partial^2}{\partial x_j\partial x_j}(F_i\hat{\mathbf{e}}_i)$$

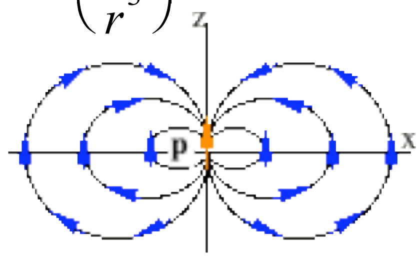
$$\underline{\nabla\times(\nabla\times\mathbf{F})} = \underline{\nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}}$$

Problem 2.8

$$\nabla\Phi = \nabla\left(\frac{\mathbf{p}\cdot\mathbf{x}}{r^3}\right) = \frac{\nabla(\mathbf{p}\cdot\mathbf{x})}{r^3} + (\mathbf{p}\cdot\hat{\mathbf{r}}\,r)\nabla\left(\frac{1}{r^3}\right)$$

$$\nabla(\mathbf{p}\cdot\mathbf{x}) = \hat{\mathbf{e}}_j\frac{\partial}{\partial x_j}p_i x_i$$

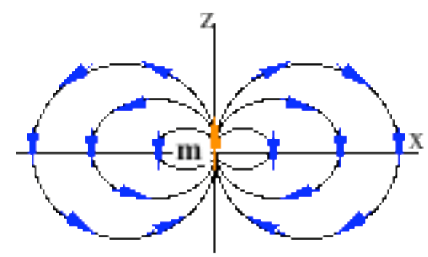
$$= \hat{\mathbf{e}}_j p_i\frac{\partial x_i}{\partial x_j} = p_i\hat{\mathbf{e}}_i = \mathbf{p}$$



$$\nabla\left(\frac{1}{r^3}\right) = -3\frac{\hat{\mathbf{r}}}{r^4}$$

$$\underline{\nabla\Phi = \frac{1}{r^3}(\mathbf{p} - 3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}})}$$

$$\nabla\times\mathbf{A} = \nabla\times\left(\frac{\mathbf{m}\times\mathbf{x}}{r^3}\right) = \frac{1}{r^3}\nabla\times(\mathbf{m}\times\mathbf{x}) + \nabla\left(\frac{1}{r^3}\right)\times(\mathbf{m}\times\mathbf{x})$$



$$[\nabla\times(\mathbf{m}\times\mathbf{x})]_i = \varepsilon_{ijk}\varepsilon_{klm}m_l\frac{\partial x_m}{\partial x_j}$$

$$= \varepsilon_{ijk}\varepsilon_{klj}m_l = (\delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl})m_l$$

$$= \delta_{jj}m_i - m_i = 3m_i - m_i = 2m_i = 2\mathbf{m}$$

$$\nabla\times\mathbf{A} = \frac{2\mathbf{m}}{r^3} - 3\frac{\hat{\mathbf{r}}\times(\mathbf{m}\times\hat{\mathbf{r}})}{r^3} = \frac{1}{r^3}[2\mathbf{m} - 3\{\mathbf{m}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}) - \hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}})\}]$$

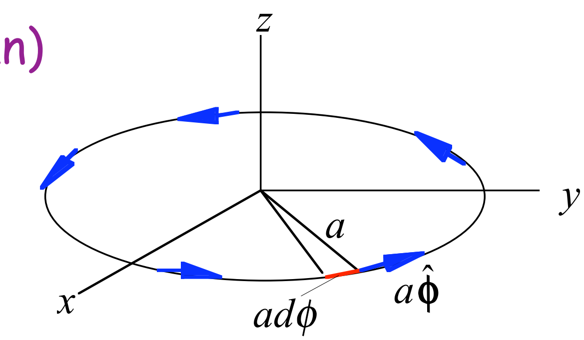
$$\underline{= \frac{1}{r^3}[-\mathbf{m} + 3\hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}})]}$$

Problem 2.9

$$\mathbf{F}(\mathbf{x}) = \hat{\mathbf{k}} \times \mathbf{x} = \hat{\mathbf{k}} \times \left(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} \right) = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$$ (Cartesian)

$$\mathbf{F}(\mathbf{x}) = \hat{\mathbf{k}} \times \mathbf{r} = r\,\hat{\mathbf{k}} \times \hat{\mathbf{r}} = r\,\hat{\boldsymbol{\phi}}$$ (Cylindrical)

$$d\ell = \hat{\boldsymbol{\phi}}\, a d\phi$$



$$\oint_C \mathbf{F} \cdot d\ell = \int_0^{2\pi} a\hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}}\, a d\phi = \underline{2\pi a^2}$$

<u>Stokes's theorem</u>

$$\oint_C \mathbf{F} \cdot d\ell = \oint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$$

$$\mathbf{F}(\mathbf{x}) = r\,\hat{\boldsymbol{\phi}}$$ Only phi component $F_\phi = r$

$$\nabla \times \mathbf{F} = \hat{\mathbf{r}}\left(-\frac{\partial F_\phi}{\partial z} \right) + \hat{\mathbf{k}}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r F_\phi \right) \right) = 2\hat{\mathbf{k}}$$

$$d\mathbf{A} = \hat{\mathbf{k}}\, r dr d\phi$$

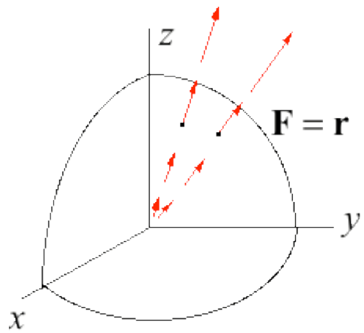
$$\oint_C \mathbf{F} \cdot d\ell = \oint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = \oint_S 2\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}\, r dr d\phi$$

$$= 2\int_0^a r dr \int_0^{2\pi} d\phi = \underline{2\pi a^2}$$

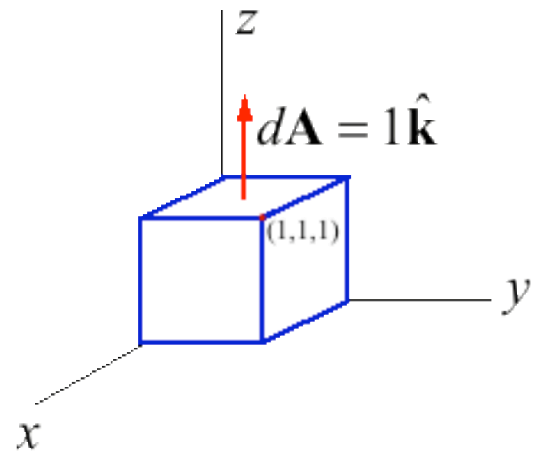
Problem 2.10

Position vector in 2 systems

$$\mathbf{F}(\mathbf{x}) = \mathbf{x} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} = r\,\hat{\mathbf{r}}$$



Back,left, and bottom faces of cube are parallel to field



Flux through the top face: $\oint_{Top} \mathbf{F}\cdot d\mathbf{A} = \mathbf{x}\cdot\hat{\mathbf{k}} = z = 1$

Total flux: $\oint_S \mathbf{F}\cdot d\mathbf{A} = \underline{3}$

Gauss's theorem: $\oint_S \mathbf{F}\cdot d\mathbf{A} = \oint_V \nabla\cdot\mathbf{F}\,dV$

Divergence of F

$$\nabla\cdot\mathbf{F} = \frac{\partial F_i}{\partial x_i} = 1+1+1 = 3$$

Volume integral

$$\oint_V \nabla\cdot\mathbf{F}\,dV = 3\oint_V dV = \underline{3}$$

Problem 2.12

$$\underline{f = x^2 + y^2 + z^2}$$

$$\nabla f = \hat{\mathbf{e}}_i \frac{\partial f}{\partial x_i} = 2\left(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}\right) = 2\mathbf{x} = \underline{2r\,\hat{\mathbf{r}}}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_i \partial x_i} = \underline{6}$$

$$\underline{f = r^2}$$

$$\nabla f = \hat{\mathbf{r}} \frac{\partial f}{\partial r} = \underline{2r\,\hat{\mathbf{r}}}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} r^2\right) = \underline{6}$$

$$\underline{f = \left(x^2 + y^2 + z^2\right)^{1/2}}$$

$$\nabla f = \hat{\mathbf{e}}_i \frac{\partial f}{\partial x_i} = \frac{\left(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}\right)}{\left(x^2 + y^2 + z^2\right)^{1/2}} = \frac{\mathbf{r}}{r} = \underline{\hat{\mathbf{r}}}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_i \partial x_i} = \frac{2}{\left(x^2 + y^2 + z^2\right)^{1/2}} = \underline{\frac{2}{r}}$$

$$\underline{f = r}$$

$$\nabla f = \frac{\partial}{\partial r}(r)\hat{\mathbf{r}} = \underline{\hat{\mathbf{r}}}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial r}{\partial r}\right) = \underline{\frac{2}{r}}$$

$$\underline{f = \left(x^2 + y^2 + z^2\right)^{-1/2}}$$

$$\nabla f = \hat{\mathbf{e}}_i \frac{\partial f}{\partial x_i} = -\frac{\left(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}\right)}{\left(x^2 + y^2 + z^2\right)^{3/2}} = \underline{-\frac{\hat{\mathbf{r}}}{r^2}}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_i \partial x_i} = (3-3)\frac{\left(x^2 + y^2 + z^2\right)}{\left(x^2 + y^2 + z^2\right)^{5/2}} = \underline{0}$$

$$\underline{f = r^{-1}}$$

$$\nabla f = \frac{\partial}{\partial r}\left(r^{-1}\right)\hat{\mathbf{r}} = \underline{-\frac{\hat{\mathbf{r}}}{r^2}}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} r^{-1}\right) = \underline{0}$$

Problem 2.13

$\underline{\mathbf{x} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}$

$$\nabla\cdot(\mathbf{x}) = \frac{\partial}{\partial x_i}(x_i) = \underline{3}$$

$$\nabla\times(\mathbf{x}) = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(x_k) = \varepsilon_{ikk} = \underline{0}$$

$\underline{\mathbf{x} = r\,\hat{\mathbf{r}}}$

$$\nabla\cdot(r\,\hat{\mathbf{r}}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\right) = \underline{3}$$

$$\nabla\times\left(r\,\hat{\mathbf{r}}\right) = \underline{0}$$

$$\underline{\mathbf{x}/r = \frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{\left(x^2+y^2+z^2\right)^{1/2}}}$$

$$\nabla\cdot(\mathbf{x}/r) = \frac{\partial}{\partial x_i}\left[\frac{x_i}{r}\right] = \underline{\frac{2}{r}}$$

$$\left[\nabla\times(\mathbf{x}/r)\right]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left[\frac{x_k}{r}\right] = -\varepsilon_{ijk}\frac{x_k x_j}{r^3} = \underline{0}$$

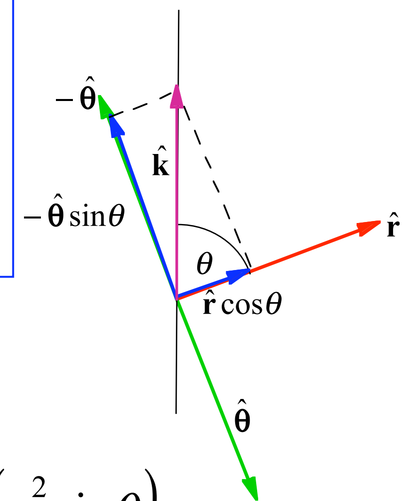
$\underline{\mathbf{x}/r = \hat{\mathbf{r}}}$

$$\nabla\cdot\hat{\mathbf{r}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\right) = \underline{\frac{2}{r}}$$

$$\nabla\times\hat{\mathbf{r}} = \underline{0}$$

|unit vectors|=1

$\underline{\hat{k}\text{ construction}}$



$\underline{\hat{\mathbf{k}}\times\mathbf{x} = \hat{\mathbf{k}}\times\left(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}\right) = x\,\hat{\mathbf{j}} - y\hat{\mathbf{i}}}$

$$\nabla\cdot\left(x\,\hat{\mathbf{j}} - y\hat{\mathbf{i}}\right) = \underline{0}$$

$$\nabla\times\left(x\,\hat{\mathbf{j}} - y\hat{\mathbf{i}}\right) = \underline{2\hat{\mathbf{k}}}$$

$\underline{\hat{\mathbf{k}}\times\mathbf{x} = r\,\hat{\mathbf{k}}\times\hat{\mathbf{r}} = r\sin\theta\,\hat{\boldsymbol{\phi}}}$

$$\nabla\cdot\left(r\sin\theta\,\hat{\boldsymbol{\phi}}\right) = \underline{0}$$

$$\nabla\times\left(r\sin\theta\,\hat{\boldsymbol{\phi}}\right) = \frac{\hat{\mathbf{r}}}{r\sin\theta}\frac{\partial}{\partial\theta}\left(r\sin^2\theta\right) - \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial r}\left(r^2\sin\theta\right)$$

$$= 2\left(\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\right) = \underline{2\hat{\mathbf{k}}}$$

Problem 2.14

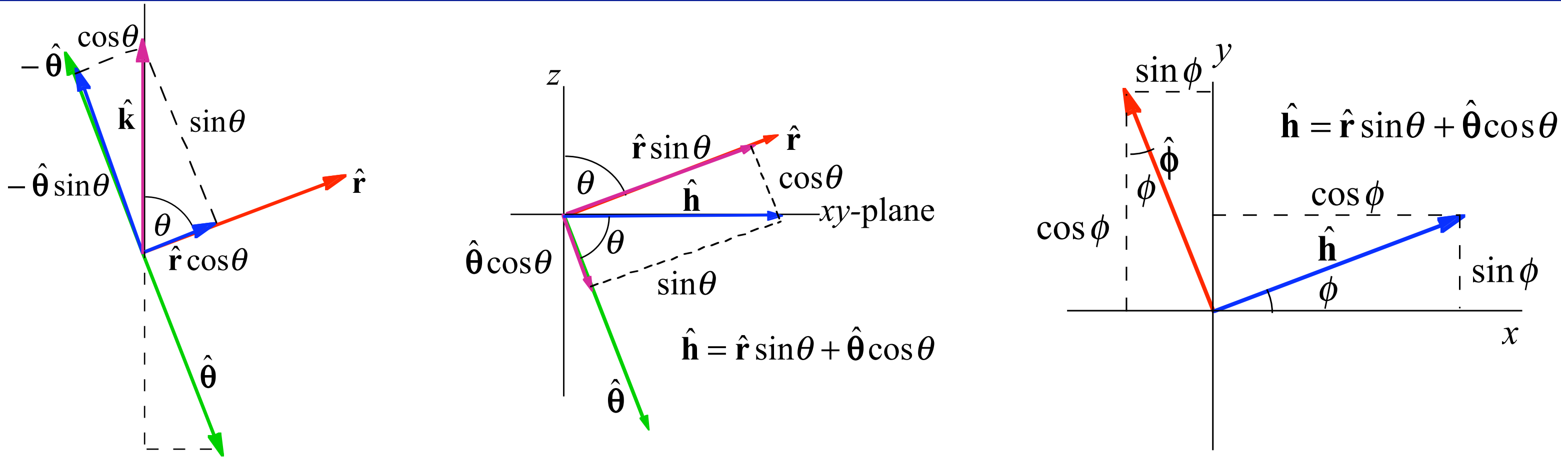


fig 1 fig 2 fig 3

see fig. 1: $\hat{\mathbf{k}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol\theta}\sin\theta$

see figs 2&3:
$$\hat{\mathbf{j}} = \hat{\mathbf{r}}\sin\theta\sin\phi + \hat{\boldsymbol\theta}\cos\theta\sin\phi + \hat{\boldsymbol\phi}\cos\phi$$
$$\hat{\mathbf{i}} = \hat{\mathbf{r}}\sin\theta\cos\phi + \hat{\boldsymbol\theta}\cos\theta\cos\phi - \hat{\boldsymbol\phi}\sin\phi$$

$$\mathbf{R} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \qquad \mathbf{R}^T = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

$$\hat{\mathbf{r}} = \hat{\mathbf{i}}\sin\theta\cos\phi + \hat{\mathbf{j}}\sin\theta\sin\phi + \hat{\mathbf{k}}\cos\theta$$
$$\hat{\boldsymbol\theta} = \hat{\mathbf{i}}\cos\theta\cos\phi + \hat{\mathbf{j}}\cos\theta\sin\phi - \hat{\mathbf{k}}\sin\theta$$
$$\hat{\boldsymbol\phi} = -\hat{\mathbf{i}}\sin\phi + \hat{\mathbf{j}}\cos\phi$$

Problem 2.19

$$\mathbf{F}(\mathbf{x}) = C\mathbf{x} = C\left(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}\right) \qquad \nabla\cdot\mathbf{F} = \frac{\partial F_i}{\partial x_i} = 3C$$

Gauss's theorem: $\oint_S \mathbf{F}\cdot d\mathbf{A} = \oint_V (\nabla\cdot\mathbf{F})\,dV = 3C(V) = \underline{3C\ell^3}$

Direct calculation: $\oint_S \mathbf{F}\cdot d\mathbf{A} = 6\oint_S \mathbf{x}\cdot\hat{\mathbf{k}}\,dA = 6Cz\int_{-\ell/2}^{\ell/2} dx \int_{-\ell/2}^{\ell/2} dy = 6C\frac{\ell}{2}\ell^2 = \underline{3C\ell^3}$

Problem 2.22

Given: $h(\mathbf{x}) = (\mathbf{x}\times\mathbf{A})\cdot(\mathbf{x}\times\mathbf{B})$ Prove: $\nabla h(\mathbf{x}) = \mathbf{A}\times(\mathbf{x}\times\mathbf{B}) + \mathbf{B}\times(\mathbf{x}\times\mathbf{A})$

Cross products: $(\mathbf{x}\times\mathbf{A})_i = \varepsilon_{ijk}x_jA_k$ $(\mathbf{x}\times\mathbf{B})_i = \varepsilon_{ilm}x_lA_m$

Collect terms: $h(\mathbf{x}) = (\mathbf{x}\times\mathbf{A})_i(\mathbf{x}\times\mathbf{B})_i = \varepsilon_{ijk}x_jA_k\varepsilon_{ilm}x_lA_m$

Gradient:
$$\left[\nabla h(\mathbf{x})\right]_n = \frac{\partial}{\partial x_n}\left[\varepsilon_{ijk}\varepsilon_{ilm}x_jA_kx_lB_m\right]$$
$$= \frac{\partial}{\partial x_n}\left[\left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right)x_jA_kx_lB_m\right]$$
$$= \frac{\partial}{\partial x_n}\left[x_jx_jA_kB_k - x_kA_kx_jB_j\right]$$
$$= 2x_nA_kB_k - A_nx_jB_j - x_kA_kB_n$$

Group terms:
$$\left[\nabla h(\mathbf{x})\right]_n = \left(x_nA_kB_k - x_kA_kB_n\right) + \left(x_nA_kB_k - A_nx_jB_j\right)$$
$$\nabla h(\mathbf{x}) = \left[\mathbf{x}(\mathbf{A}\cdot\mathbf{B}) - \mathbf{B}(\mathbf{A}\cdot\mathbf{x})\right] + \left[\mathbf{x}(\mathbf{B}\cdot\mathbf{A}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{x})\right]$$

$$\underline{\nabla h(\mathbf{x}) = \left[\mathbf{A}\times(\mathbf{x}\times\mathbf{B})\right] + \left[\mathbf{B}\times(\mathbf{x}\times\mathbf{A})\right]}$$

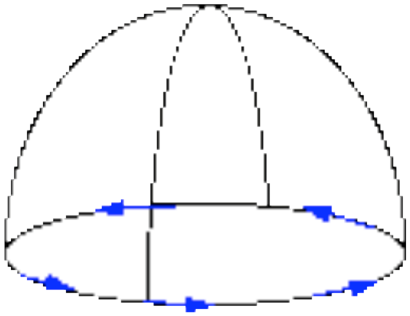
Triple cross product

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$$
$$\mathbf{A}\times(\mathbf{x}\times\mathbf{B}) = \mathbf{x}(\mathbf{A}\cdot\mathbf{B}) - \mathbf{B}(\mathbf{A}\cdot\mathbf{x})$$
$$\mathbf{B}\times(\mathbf{x}\times\mathbf{A}) = \mathbf{x}(\mathbf{B}\cdot\mathbf{A}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{x})$$

Problem 2.24

Line integral: $\oint_C \mathbf{F}\cdot d\ell = r\int_0^{2\pi} d\phi = \underline{2\pi r}$

$\underline{\mathbf{F} = \hat{\boldsymbol{\phi}}}$



Stokes's theorem: $\oint_C \mathbf{F}\cdot d\ell = \oint_H (\nabla\times\mathbf{F})\cdot d\mathbf{A}$

Curl of F:

$$\nabla\times\mathbf{F} = \nabla\times\hat{\boldsymbol{\phi}} = \frac{\hat{\mathbf{r}}}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta\right) + \frac{\hat{\boldsymbol{\theta}}}{r}\left(-\frac{\partial}{\partial r}r\right)$$

$$= \frac{\hat{\mathbf{r}}\cos\theta}{r\sin\theta} - \frac{\hat{\boldsymbol{\theta}}}{r} = \frac{1}{r\sin\theta}\left(\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\right)$$

Integrand: $d\mathbf{A} = \hat{\mathbf{r}}\, r^2 \sin\theta d\theta d\phi$

$$(\nabla\times\mathbf{F})\cdot d\mathbf{A} = \frac{1}{r\sin\theta}\left(\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\right)\cdot\left(\hat{\mathbf{r}}\, r^2\sin\theta d\theta d\phi\right)$$

$$= r\cos\theta d\theta d\phi$$

<u>Hemisphere Surface integral</u>

$$\oint_H (\nabla\times\mathbf{F})\cdot d\mathbf{A} = r\int_0^{\pi/2}\cos\theta d\theta\int_0^{2\pi} d\phi$$

$$= 2\pi r\sin\theta\,\Big|_0^{\pi/2} = \underline{2\pi r}$$

<u>Disk surface integral</u>

when $\theta = \pi/2,\ \hat{\mathbf{k}} = -\hat{\boldsymbol{\theta}}$

$$d\mathbf{A} = -\hat{\boldsymbol{\theta}}\, r' dr' d\phi$$

$$(\nabla\times\mathbf{F})\cdot d\mathbf{A} = dr' d\phi$$

$$\oint_D (\nabla\times\mathbf{F})\cdot d\mathbf{A} = \int_0^r dr'\int_0^{2\pi} d\phi = \underline{2\pi r}$$

Problem 2.27

$$\underline{\mathbf{F}(\mathbf{x}) = \mathbf{x}/r^3 = \hat{\mathbf{r}}/r^2}$$

Integral over sphere radius a

$$d\mathbf{A} = \hat{\mathbf{r}}\, r^2 \sin\theta d\theta d\phi$$

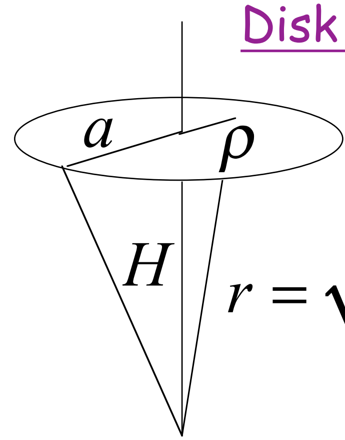
$$\oint_S \mathbf{F}\cdot d\mathbf{A} = \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = 2\pi\left(-\cos\theta\right)\Big|_0^{\pi} = \underline{4\pi}$$

Prep for integral over Disk

$$d\mathbf{A} = \hat{\mathbf{k}}\,\rho d\rho d\phi = \left(\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\right)\rho d\rho d\phi$$

$$r = \sqrt{H^2+\rho^2}; \quad \mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\hat{\mathbf{r}}}{H^2+\rho^2}; \quad \cos\theta = \frac{H}{\sqrt{H^2+\rho^2}}$$

Disk radius a



$$r = \sqrt{H^2+\rho^2}$$

Integral over Disk radius a

$$\oint_S \mathbf{F}\cdot d\mathbf{A} = H\int_0^{a} \frac{\rho}{\left(H^2+\rho^2\right)^{3/2}} d\rho \int_0^{2\pi} d\phi$$

$$= \underline{2\pi\left(1 - \frac{H}{\left(H^2+a^2\right)^{1/2}}\right)}$$

Integral when H = a

$$\oint_S \mathbf{F}\cdot d\mathbf{A} = \underline{2\pi\left(1-\frac{1}{\sqrt{2}}\right)}$$