PHY481: Electromagnetism

Curl and divergence of E

& a bit more vector mathematics

Problem 2.27

$$\underline{\mathbf{F}(\mathbf{x}) = \mathbf{x}/r^3 = \hat{\mathbf{r}}/r^2}$$

Integral over sphere radius a

$$d\mathbf{A} = \hat{\mathbf{r}}\, r^2 \sin\theta\, d\theta\, d\phi$$

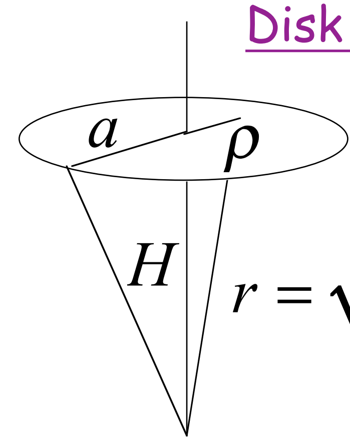
$$\oint_S \mathbf{F}\cdot d\mathbf{A} = \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = 2\pi(-\cos\theta)\Big|_0^{\pi} = \underline{4\pi}$$

Prep for integral over Disk

$$d\mathbf{A} = \hat{\mathbf{k}}\,\rho\, d\rho\, d\phi = (\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)\,\rho\, d\rho\, d\phi$$

$$r = \sqrt{H^2+\rho^2}; \quad \mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\hat{\mathbf{r}}}{H^2+\rho^2}; \quad \cos\theta = \frac{H}{\sqrt{H^2+\rho^2}}$$

Disk radius a



$$r = \sqrt{H^2+\rho^2}$$

Integral over Disk radius a

$$\oint_S \mathbf{F}\cdot d\mathbf{A} = H\int_0^{a} \frac{\rho}{(H^2+\rho^2)^{3/2}}\, d\rho \int_0^{2\pi} d\phi$$

$$= \underline{2\pi\left(1 - \frac{H}{(H^2+a^2)^{1/2}}\right)}$$

Integral when H = a

$$\oint_S \mathbf{F}\cdot d\mathbf{A} = \underline{2\pi\left(1 - \frac{1}{\sqrt{2}}\right)}$$

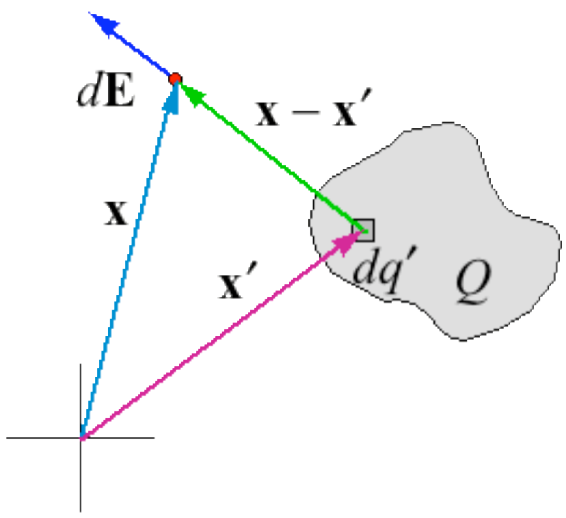
Electric field of charge distributions

- Charge distributions

	Volume	Surface	Line
Distribution:	Volume	Surface	Line
Charge density:	$\rho(\mathbf{x}')$	$\sigma(\mathbf{x}')$	$\lambda(\mathbf{x}')$
Total Charge:	$Q = \int \rho(\mathbf{x}')d^3x'$	$Q = \int \sigma(\mathbf{x}')d^2x'$	$Q = \int \lambda(\mathbf{x}')dx'$

- Integration over charge distribution

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x'$$

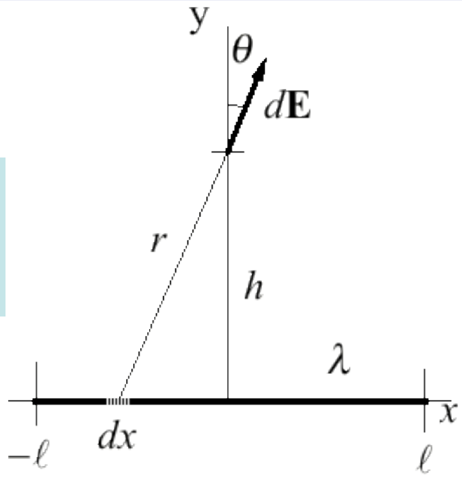


Old way   Line of charge   New way

$$r = \left(h^2 + x^2\right)^{1/2}$$

$$\cos\theta = \frac{h}{\left(h^2 + x^2\right)^{1/2}}$$

$$dq = \lambda dx$$

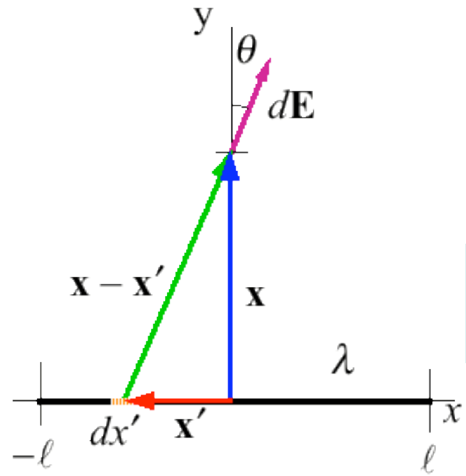


$$E_y = \int dE_y = \int dE\cos\theta = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}\cos\theta$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{-\ell}^{\ell} \frac{\lambda dx}{\left(h^2 + x^2\right)}\cos\theta$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{-\ell}^{\ell} \frac{\lambda dx}{\left(h^2 + x^2\right)}\left(\frac{h}{\left(h^2 + x^2\right)^{1/2}}\right)$$

$$E_y = \frac{\lambda h}{4\pi\varepsilon_0}\int_{-\ell}^{\ell}\left(h^2 + x^2\right)^{-3/2} dx$$



$$\mathbf{x} = h\,\hat{\mathbf{j}}$$

$$\mathbf{x}' = x'\,\hat{\mathbf{i}}$$

$$\mathbf{x} - \mathbf{x}' = h\,\hat{\mathbf{j}} - x'\,\hat{\mathbf{i}}$$

$$\left|\mathbf{x} - \mathbf{x}'\right| = \left(h^2 + x'^2\right)^{1/2}$$

$$\rho(\mathbf{x}')d^3x' = \lambda dx'$$

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{x} - \mathbf{x}'}{\left|\mathbf{x} - \mathbf{x}'\right|^3}\rho(\mathbf{x}')d^3x'$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{-\ell}^{\ell} \frac{h\,\hat{\mathbf{j}} - x'\,\hat{\mathbf{i}}}{\left(h^2 + x'^2\right)^{3/2}}\lambda dx'$$

$$\mathbf{E} = \hat{\mathbf{j}}\frac{h\lambda}{4\pi\varepsilon_0}\int_{-\ell}^{\ell}\left(h^2 + x'^2\right)^{-3/2} dx'$$

Angular dependence automatically included !

Curl and divergence of E

- Prove differential form of Gauss's Law

Gauss's Law: $\nabla \cdot \mathbf{E}(\mathbf{x}) = \dfrac{\rho(\mathbf{x})}{\varepsilon_0}$ True even for $\mathbf{E}(\mathbf{x},t)$

with

$\nabla \times \mathbf{E}(\mathbf{x}) = 0$ True only for static charge

- Will need definition of the scalar potential function $V(\mathbf{x})$

$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

$\mathbf{x}$ = evaluation coordinates

$\mathbf{x}'$ = source positions

$|\,\mathbf{x} - \mathbf{x}'\,|$ (not $|\;\;|^2$ or $|\;\;|^3$)

- Will need Gradient of V
  - Derivatives are with respect to x, not x': $\nabla = \hat{\mathbf{e}}_i \dfrac{\partial}{\partial x_i}$
  - Bring Del into the integral over d^3x'

$$\nabla V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\nabla\left(\int \frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}\right) = \frac{1}{4\pi\varepsilon_0}\int \rho(\mathbf{x}')\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)d^3x'$$

Gradient of 1/|x-x'|

Since:

$$\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)=\frac{-1}{|\mathbf{x}-\mathbf{x}'|^2}\nabla|\mathbf{x}-\mathbf{x}'|$$
$$=-\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}$$

Using chain rule

$$\frac{\partial}{\partial x_i}f[g(x)]=f'[g(x)]\frac{\partial g(x)}{\partial x_i}$$

$$|\mathbf{x}-\mathbf{x}'|=\sqrt{(x_i-x_i')^2}$$
$$\nabla|\mathbf{x}-\mathbf{x}'|=\frac{2(x_i-x_i')\,\mathbf{e}_i}{2\sqrt{(x_i-x_i')^2}}=\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}$$

, then

$$\nabla V(\mathbf{x})=\frac{1}{4\pi\varepsilon_0}\int\nabla\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\rho(\mathbf{x}')d^3x'$$
$$=-\frac{1}{4\pi\varepsilon_0}\int\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x'$$
$$=-\mathbf{E}(\mathbf{x})$$

Presented as

$$\mathbf{E}(\mathbf{x})=-\nabla V(\mathbf{x})$$

Del squared of V

- Still trying to prove Gauss's Law !

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\varepsilon_0}$$

- We now have available $\mathbf{E}(\mathbf{x}) = -\nabla V(\mathbf{x})$

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \nabla \cdot \left(-\nabla V\right) = -\nabla^2 V$$

(-) Laplacian of V

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = -\nabla^2 V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int -\nabla^2\left[\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]\rho(\mathbf{x}')d^3x'$$

- Two will be the same if we can show that

$$\int_{Vol.} -\nabla^2\left[\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]d^3x' = 4\pi \text{ if } Volume \text{ includes } \mathbf{x}' = \mathbf{x}$$

Involves Dirac's delta function

Laplacian of $1/|\mathbf{x}-\mathbf{x}'|$

- To prove Gauss's Law we need to show

$$\int_{Vol.} -\nabla^2 \left[\frac{1}{|\mathbf{x}-\mathbf{x}'|} \right] d^3x' = 4\pi \text{ if } Volume \text{ includes } \mathbf{x}' = \mathbf{x}$$

- Point source at $\mathbf{x}' = 0$, and use spherical coordinates $\mathbf{x} = \mathbf{r}$

$$\nabla^2 \left[\frac{1}{|\mathbf{x}|} \right] = \nabla^2 \left[\frac{1}{r} \right] = \frac{d^2}{dr^2} \left[\frac{1}{r} \right] + \frac{2}{r} \frac{d}{dr} \left[\frac{1}{r} \right]$$

$$= \frac{2}{r^3} - \frac{2}{r^3} \quad = 0 \text{ everywhere ?}$$

What about at $r = 0$?

- Strategy to evaluate at the singular point, $r = 0$

Use Gauss's theorem & a little sphere radius R around $r = 0$

Strange Laplacian of 1/|**x**-**x**′|

- Value of $\nabla^2\left[\frac{1}{r}\right]=0$ But perhaps not at r = 0 !
- Use Gauss's theorem to evaluate at r = 0

$$\int_V \nabla\cdot\mathbf{F}\,d^3x=\int_S \mathbf{F}\cdot d\mathbf{A}$$

with $\mathbf{F}=\nabla\left[1/r\right]$

$$\int_V \nabla^2\left[\frac{1}{r}\right]d^3x=\int_S \nabla\left[\frac{1}{r}\right]\cdot\hat{\mathbf{r}}\,dA$$

$$\nabla\left[\frac{1}{r}\right]=\frac{\partial}{\partial r}\left[\frac{1}{r}\right]\hat{\mathbf{r}}=\frac{-1}{r^2}\hat{\mathbf{r}}$$

$$dA=R^2\sin\theta\,d\theta\,d\phi$$

- Evaluate right side with sphere, radius R around origin

$$\int_V \nabla^2\left[\frac{1}{r}\right]d^3x=\frac{-1}{R^2}\int_0^{\pi} R^2\sin\theta\,d\theta\int_0^{2\pi} d\phi=-4\pi$$

(pretty amazing)

Does not depend on sphere radius R !

$$\int_{Vol.} -\nabla^2\left[\frac{1}{r}\right]d^3x'=4\pi \text{ if } Volume \text{ includes } r=0$$

Dirac delta function

- Strange behavior of the Laplacian of 1/r

$$\int_{Vol.} -\nabla^2\left[\frac{1}{|\mathbf{x}|}\right] d^3x' = \begin{cases} 4\pi & \text{if } Volume \text{ includes } \mathbf{x}=0 \\ 0 & \text{elsewhere} \end{cases}$$

- Define Dirac delta function (very useful in advanced physics)

$$\delta^3(\mathbf{x}) = -\frac{1}{4\pi}\nabla^2\left[\frac{1}{|\mathbf{x}|}\right]$$

$$\int_{Vol.} \delta^3(\mathbf{x}) d^3x' = \begin{cases} 1 & \text{if } Volume \text{ includes } \mathbf{x}=0 \\ 0 & \text{elsewhere} \end{cases}$$

- Finally, we prove Gauss's Law

$$\nabla\cdot\mathbf{E} = -\nabla^2 V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int -\nabla^2\left[\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]\rho(\mathbf{x}')d^3x'$$

delta function picks out value of ρ at point where $\mathbf{x} = \mathbf{x}'$

$$\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\int \delta^3(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x}')d^3x' = \frac{\rho(\mathbf{x})}{\varepsilon_0}$$

Green's functions

- Poisson's equation and a general solution

$$-\nabla^2 V(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\varepsilon_0} \qquad V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

- Green's function of a differential operator
  - The Laplacian is just one of many important differential operators
  - Consider a unit point source at **x**' $\quad \rho_0(\mathbf{x}) = \delta^3(\mathbf{x}-\mathbf{x}')$
  - The Green's function *G* for the Laplacian (which we have just found) is the solution for the potential due a unit point source

$$-\nabla^2 G\left(\mathbf{x}-\mathbf{x}'\right) = \delta^3(\mathbf{x}-\mathbf{x}') \qquad G(\mathbf{x}-\mathbf{x}') = \frac{1}{4\pi\,|\,\mathbf{x}-\mathbf{x}'\,|}$$

- The potential *V* (**x**) for an arbitrary source distribution is then

$$V(\mathbf{x}) = \int G(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x}')d^3x'$$