PHY481: Electromagnetism

Dirac delta function

E field near a boundary

Hints for solving HW problems

Dirac delta function

- P3.17c. Prove: $$\int_{-\infty}^{\infty} \delta^3(\mathbf{Ax}+\mathbf{b}) f(\mathbf{x}) d^3x = f(-\mathbf{A}^{-1}\mathbf{b}) / |\det \mathbf{A}|$$

Change variables

$$\mathbf{x}' = \mathbf{Ax} + \mathbf{b}; \quad x'_i = A_{ij} x_j + b_j$$

$$\mathbf{x} = -\mathbf{A}^{-1}\mathbf{b} \text{ when } \mathbf{x}' = 0$$

Jacobian

$$d^3x' = J d^3x \quad \text{where } J = \left| \det \left[\frac{\partial x'_i}{\partial x_k} \right] \right|$$

$$J = \left| \det \left(A_{ij} \frac{\partial x_j}{\partial x_k} \right) \right| = \left| \det A_{ij} \right|$$

Another Jacobian example:

$$d^3x = dxdydz = J\, dr d\theta d\phi$$

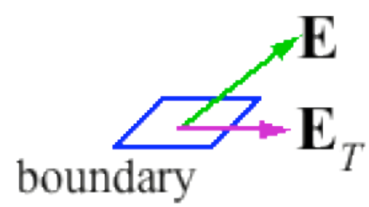
$$(x, y, z) = \left(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta \right)$$

$$J = \left| \det \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix} \right| = r^2 \sin\theta$$

Stokes's theorem and E at a boundary

- Stokes's theorem

$$\oint_S (\nabla \times \mathbf{E}) \cdot \hat{\mathbf{n}}\, dS = \oint_C \mathbf{E} \cdot d\boldsymbol{\ell}$$



$\mathbf{E}_T$ is tangential projection of **E**

- Electric field has zero curl

$$\nabla \times \mathbf{E} = 0$$

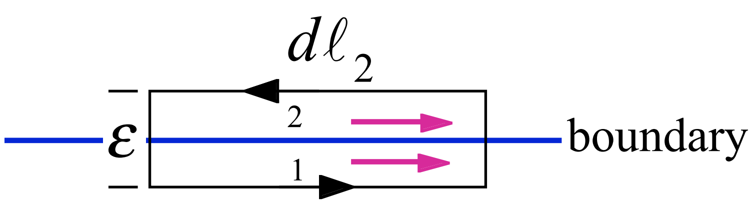
- Loop near a boundary

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0$$

$$\mathbf{E}_{T1} \cdot d\boldsymbol{\ell}_1 + \mathbf{E}_{T2} \cdot d\boldsymbol{\ell}_2 = 0$$

$$(\mathbf{E}_{T1} - \mathbf{E}_{T2}) \cdot d\boldsymbol{\ell}_1 = 0$$

$$\mathbf{E}_{T1} = \mathbf{E}_{T2}$$



$$\varepsilon \rightarrow 0$$

$$d\boldsymbol{\ell}_2 \rightarrow -d\boldsymbol{\ell}_1$$ (vectors)

Tangential components of E are continuous

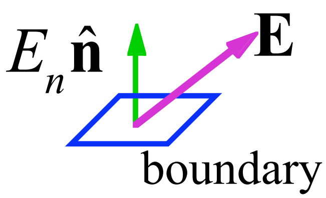
Gauss's theorem and E at a boundary

- Gauss's theorem $\oint_S \mathbf{E}\cdot d\mathbf{A} = \oint_V \nabla\cdot\mathbf{E}\, d^3x$ & law $\nabla\cdot\mathbf{E} = \rho/\varepsilon_0$

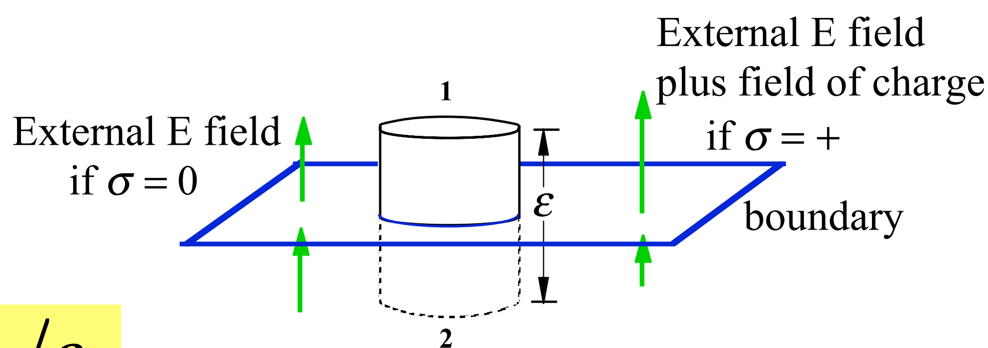
Integral form $\oint_S \mathbf{E}\cdot d\mathbf{A} = q_{encl}/\varepsilon_0$

$$\frac{1}{\varepsilon_0}\oint_V \rho(\mathbf{x})\, d^3x = \frac{q_{encl}}{\varepsilon_0}$$

E_n is normal component of **E**



- Crossing a boundary



upper disk lower disk

$$\left(\mathbf{E}_1\cdot\hat{\mathbf{n}}_1 dA_1 + \mathbf{E}_2\cdot\hat{\mathbf{n}}_2 dA_2\right) = dq_{encl}/\varepsilon_0$$

$$\left(\mathbf{E}_1 - \mathbf{E}_2\right)\cdot\hat{\mathbf{n}}_1 dA =$$

$$\left(E_{n1} - E_{n2}\right) = \sigma/\varepsilon_0$$

$\varepsilon \to 0$

$\hat{\mathbf{n}}_2 = -\hat{\mathbf{n}}_1$

$\sigma = dq_{encl} / dA$

Normal components of E differ by σ/ε_0

But if $\sigma = 0$, E_n is continuous

Problem 3.5

Vertical line of charge. Find electric field on z-axis above the charge.

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x'$$

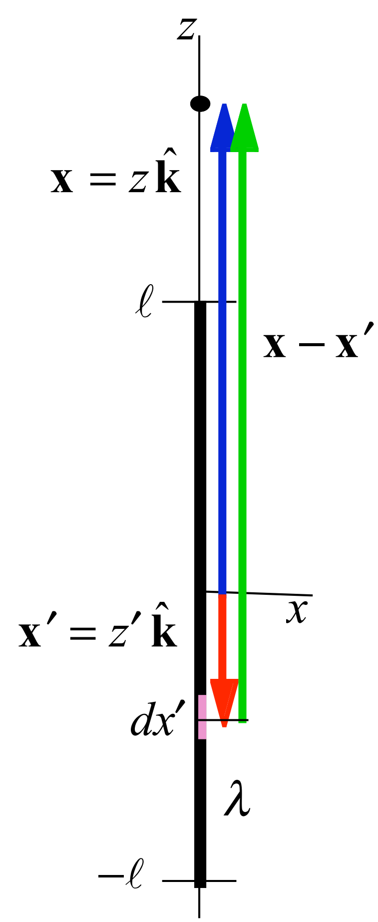
$$\mathbf{x} = z\hat{\mathbf{k}}$$

$$\mathbf{x}' = z'\hat{\mathbf{k}}$$

$$\mathbf{x}-\mathbf{x}' = (z-z')\hat{\mathbf{k}}$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{-\ell}^{\ell} \frac{(z-z')\hat{\mathbf{k}}}{(z-z')^3}\lambda dz'$$

$$\rho(\mathbf{x}')d^3x' = \lambda dz'$$



Problem 3.6

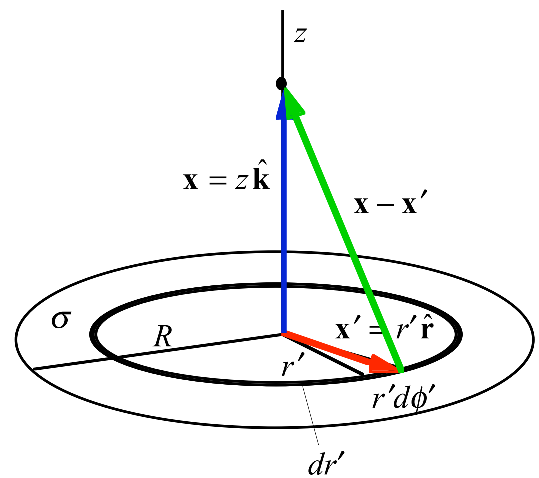
Disk of charge. Find electric field on z-axis.

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x'$$

$$= \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi \int_0^R \frac{z\hat{\mathbf{k}} - r'\hat{\mathbf{r}}}{(z^2 + r'^2)^{3/2}} r'dr'$$

$$\mathbf{x} = z\hat{\mathbf{k}}$$

$$\mathbf{x}' = r'\hat{\mathbf{r}}$$



$$\mathbf{x}-\mathbf{x}' = z\hat{\mathbf{k}} - r'\hat{\mathbf{r}}$$

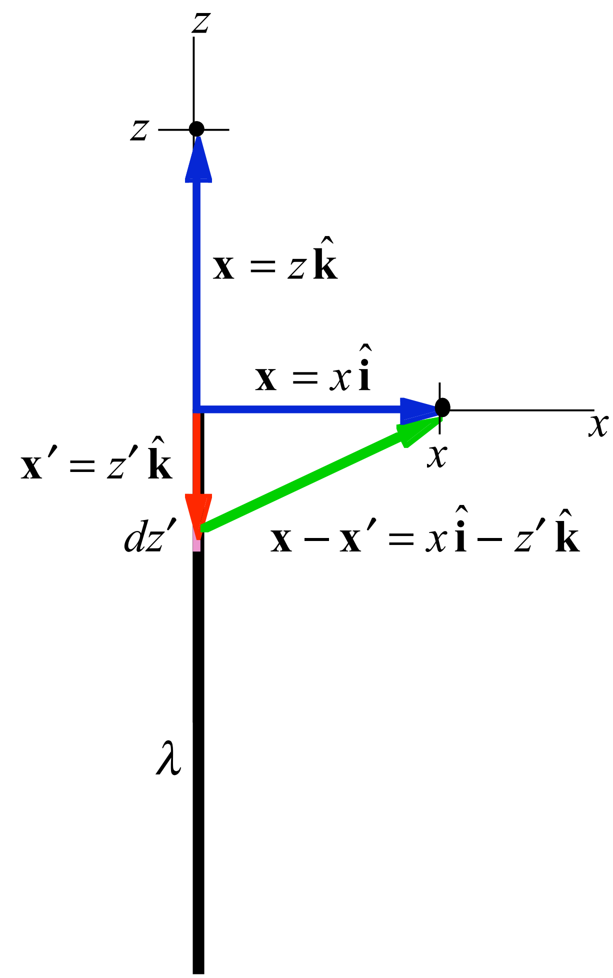
$$|\mathbf{x}-\mathbf{x}'| = \left(z^2 + r'^2\right)^{1/2}$$

$$\rho(\mathbf{x}')d^3x' = \sigma\, r'dr'd\phi'$$

Problem 3.8

Infinite line charge on negative part of z axis. Find electric field on positive part of the z-axis.

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x'$$



Problem 3.10

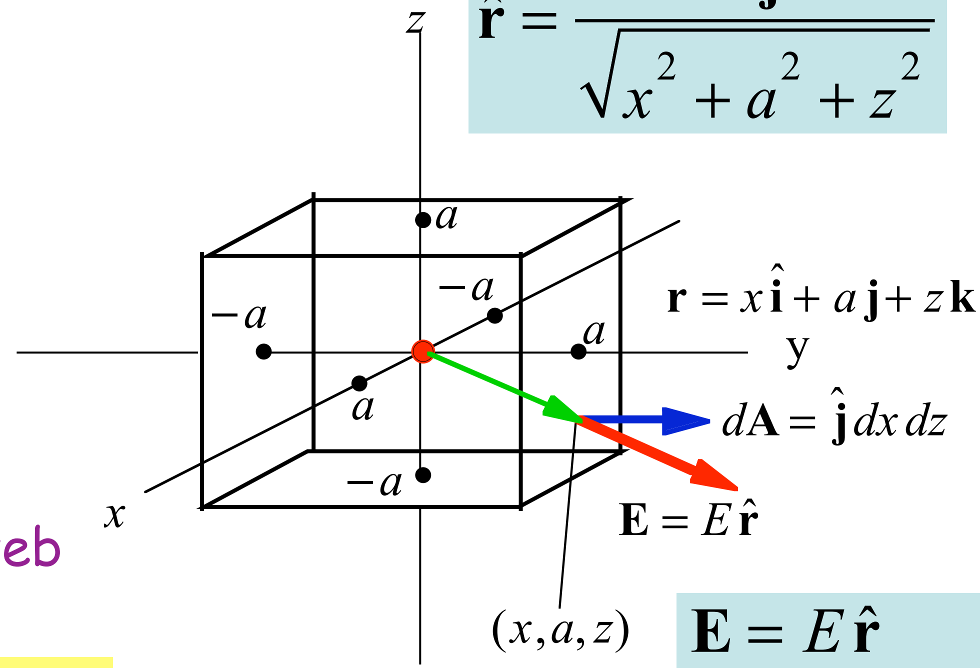
Charge q at the center of a box edge = $2a$. Check Gauss's law.

Gauss's law

$$\int_S \mathbf{E}\cdot d\mathbf{A} = \frac{q_{encl}}{\varepsilon_0}$$

$$\hat{\mathbf{r}} = \frac{x\,\hat{\mathbf{i}} + a\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{\sqrt{x^2 + a^2 + z^2}}$$

Add up the flux through the 6 faces and check it adds up to the expected value



You will need to look on the web for this integral

$$\int \frac{dx}{(a^2 + x^2)(2a^2 + x^2)} = \arctan[\quad]$$

$$\mathbf{E} = E\,\hat{\mathbf{r}}$$

$$E = \frac{q}{4\pi\varepsilon_0 r^2}$$

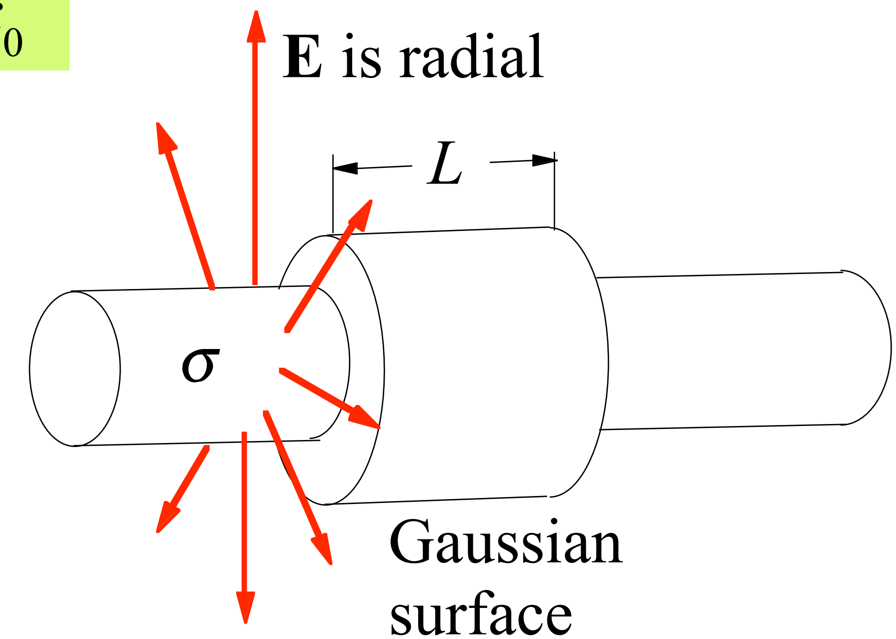
3.14

No help needed for this one!

3.15

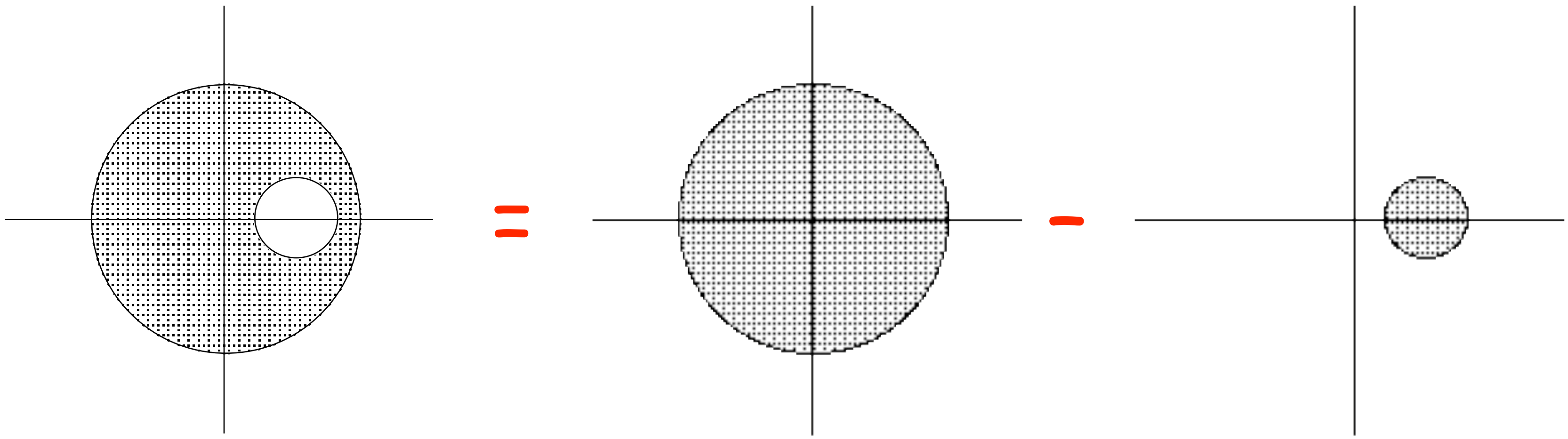
Cylinder with uniform surface charge density. Use Gauss's law to determine the radial dependence of the field.

Gauss's law $\int_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{encl}}{\varepsilon_0}$



3.42

Solid sphere with uniform charge density has a smaller sphere hollowed out. Find field in the cavity.



Determine the field inside a solid sphere using spherical coordinates.
Expect a horizontal field in the cavity, so do the subtraction using Cartesian coordinates.