PHY481: Electromagnetism

Potential $V(\mathbf{x})$

Potential energy U

Potential energy density $u(\mathbf{x})$

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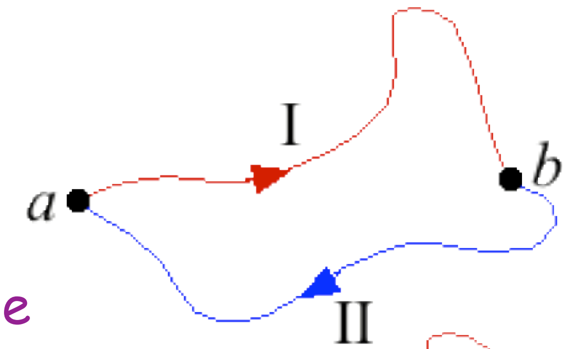
Electric potential differences

Stokes's theorem: $\oint_C \mathbf{E}\cdot d\ell = \oint_S (\nabla\times\mathbf{E})\cdot\hat{\mathbf{n}}\,dA$ but $\nabla\times\mathbf{E}=0$

Integral over closed path: $\oint_C \mathbf{E}\cdot d\ell = 0$

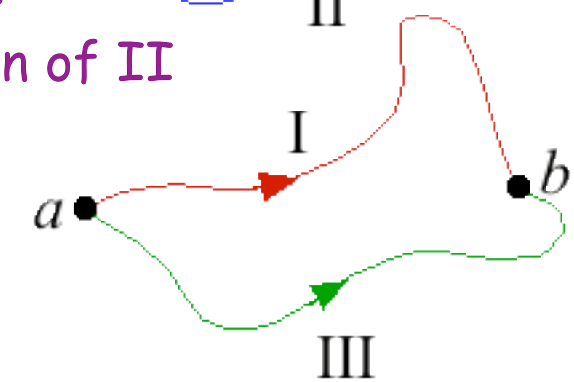
Integral around loop

$$\int_a^b \mathbf{E}\cdot d\ell_I + \int_b^a \mathbf{E}\cdot d\ell_{II} = 0$$



Reverse direction of II

$$\int_a^b \mathbf{E}\cdot d\ell_I = -\int_b^a \mathbf{E}\cdot d\ell_{II} = \int_a^b \mathbf{E}\cdot d\ell_{III}$$



Integral is independent of path

Integral depends only on end points

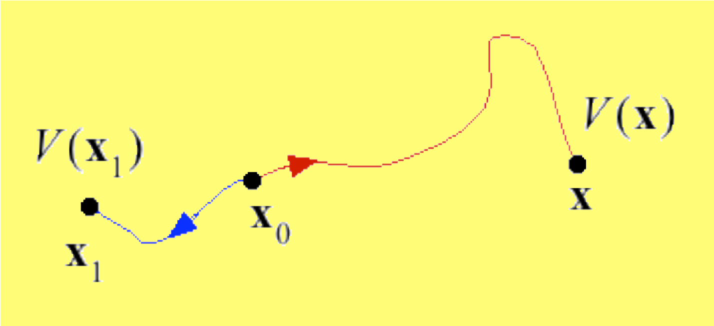
Let V(x) be "potential" at x

$$V(\mathbf{x}_b) - V(\mathbf{x}_a) = -\int_a^b \mathbf{E}\cdot d\ell$$

Integral is work done (by an external agent) in moving a unit charge from a to b.

V defined up to an additive constant

V(x) and $V(x_1)$ are potentials relative to potential at point x_0



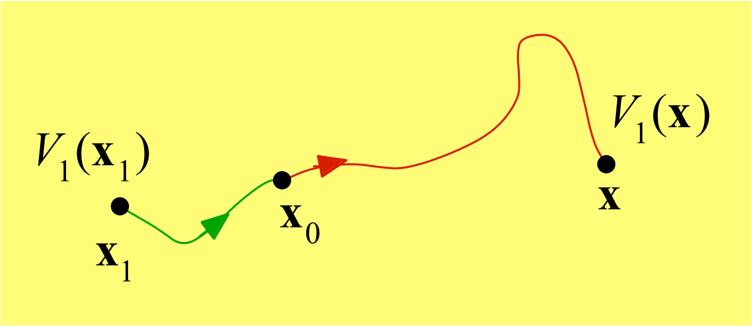
$$V(\mathbf{x}) = -\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{E}\cdot d\ell$$

$$\mathbf{x}_0 \to \mathbf{x}$$

$$V(\mathbf{x}_1) = -\int_{\mathbf{x}_0}^{\mathbf{x}_1} \mathbf{E}\cdot d\ell$$

$$\mathbf{x}_0 \to \mathbf{x}_1$$

Find $V_1(x)$: potential at point x relative to potential at point x_1



$$V_1(\mathbf{x}) = -\int_{\mathbf{x}_1}^{\mathbf{x}} \mathbf{E}\cdot d\ell \qquad \mathbf{x}_1 \to \mathbf{x}_0 \to \mathbf{x}$$

$$= \left(-\int_{\mathbf{x}_1}^{\mathbf{x}_0} \mathbf{E}\cdot d\ell\right) + \left(-\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{E}\cdot d\ell\right)$$

$$V_1(\mathbf{x}) = \quad -V(\mathbf{x}_1) \quad + \quad V(\mathbf{x})$$

Changing reference point results in a constant added to V

$$C = -V(\mathbf{x}_1)$$

Potential and field of a point charge

- Relationship between E and V

$$\mathbf{E}(\mathbf{x}) = -\nabla V(\mathbf{x})$$ where $$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

Charge density for a point charge q at origin

$$\rho(\mathbf{x}') = q\delta^3(\mathbf{x}')$$

- Potential of point charge *q* at the origin

$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|} = \frac{q}{4\pi\varepsilon_0}\int \frac{\delta^3(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

$$V(r) = \frac{q}{4\pi\varepsilon_0|\mathbf{x}|} = \frac{q}{4\pi\varepsilon_0 r} \qquad |\mathbf{x}| = r$$

- Check field is correct

$$\mathbf{E}(r) = -\nabla V(r) = -\frac{\partial V(r)}{\partial r}\hat{\mathbf{r}}$$

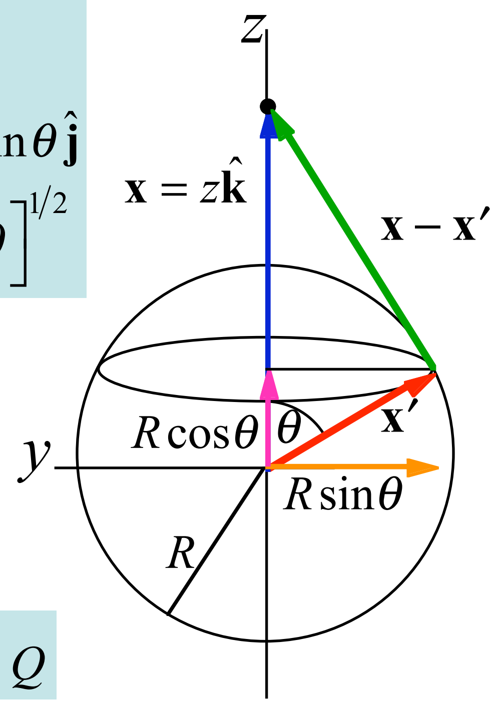
$$= -\frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\left[\frac{1}{r}\right]\hat{\mathbf{r}} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$ OK !

Potential due to spherical shell of charge

- Spherical shell, charge Q, radius R, at distance z (or r)

$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

$$\rho(\mathbf{x}')d^3x' = \sigma R^2 \sin\theta\, d\theta\, d\phi$$
$$\mathbf{x}' = R\cos\theta\,\hat{\mathbf{k}} + R\sin\theta\,\hat{\mathbf{j}}$$
$$\mathbf{x}-\mathbf{x}' = (z - R\cos\theta)\hat{\mathbf{k}} - R\sin\theta\,\hat{\mathbf{j}}$$
$$|\mathbf{x}-\mathbf{x}'| = \left[z^2 + R^2 - 2zR\cos\theta\right]^{1/2}$$



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$$V(z) = \frac{\sigma R^2}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \left[z^2 + R^2 - 2zR\cos\theta\right]^{-1/2}$$

$$= \frac{2\pi\sigma R}{4\pi\varepsilon_0 z}\left[(z+R) \mp (z-R)\right]$$

$(-)\ z > R$
$(+)\ z < R$

$$4\pi R^2\sigma = Q$$

Point charge Potential Outside

$$V(z) = \frac{Q}{4\pi\varepsilon_0 z},\ z > R$$

Constant Potential Inside

$$V(z) = \frac{Q}{4\pi\varepsilon_0 R},\ z < R$$

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ENTER ANY FUNCTION

choose a random example >>

$\int$ Sin[x]/(a^2 + b^2 + 2 a b Cos[x])^(1/2) dx

COMPUTE

TraditionalForm | StandardForm | InputForm | OutputForm

$$\int \frac{\sin(x)}{\sqrt{a^2 + 2\,b\cos(x)\,a + b^2}}\,dx =$$

$$-\frac{\sqrt{a^2 + 2\,b\cos(x)\,a + b^2}}{a\,b}$$

Time to calculate: 0.05 second

Potential for sphere with uniform charge density

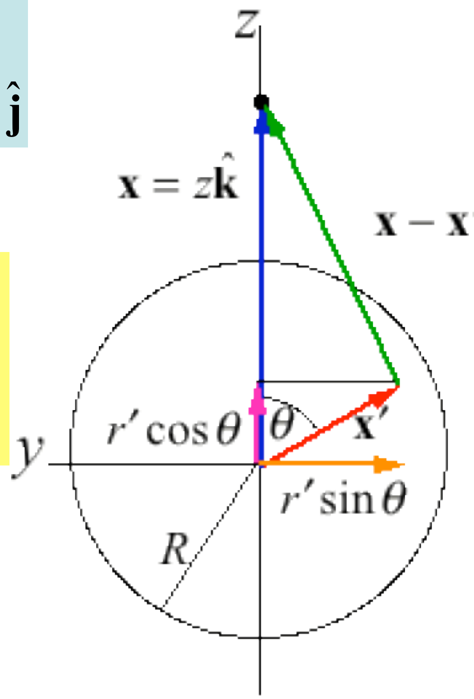
- Sphere, uniform density ρ, radius R, at distance z (or r)

$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

$$\mathbf{x}' = r'\cos\theta\,\hat{\mathbf{k}} + r'\sin\theta\,\hat{\mathbf{j}}$$

$$\mathbf{x}-\mathbf{x}' = (z - r'\cos\theta)\hat{\mathbf{k}} - r'\sin\theta\,\hat{\mathbf{j}}$$

$$\rho(\mathbf{x}')d^3x' = \rho r'^2 dr' \sin\theta\, d\theta\, d\phi$$



$$V(z) = \frac{2\pi\rho}{4\pi\varepsilon_0}\int_0^R r'^2 dr' \int_0^\pi \sin\theta d\theta \left[z^2 + r'^2 - 2zr'\cos\theta\right]^{-1/2}$$

$$= \frac{2\pi\rho}{4\pi\varepsilon_0 z}\int_0^R r'dr'\left\{\left[(z+r')^2\right]^{1/2} - \left[(z-r')^2\right]^{1/2}\right\}$$

- Outside the sphere $z > R$

$$4\pi R^3\rho/3 = Q$$

$$V(z) = \frac{2\pi\rho}{4\pi\varepsilon_0 z}\int_0^R r'dr'\left\{(z+r')-(z-r')\right\} = \frac{4\pi\rho}{4\pi\varepsilon_0 z}\left[\frac{r'^3}{3}\right]_0^R = \frac{Q}{4\pi\varepsilon_0 z}$$

Uniformly charged sphere (cont'd)

- Inside the sphere

$$V(z) = \frac{2\pi\rho}{4\pi\varepsilon_0 z}\int_0^R r'dr'\left\{\left[(z+r')^2\right]^{1/2} - \left[(z-r')^2\right]^{1/2}\right\}$$

$$= \frac{2\pi\rho}{4\pi\varepsilon_0 z}\int_0^R r'dr'\left\{(z+r') \pm (z-r')\right\}$$

Need positive root

Split integral into two parts $0 < r' < z$ & $z < r' < R$

$r' < z$ use $-$ $\quad$ $r' > z$ use $+$

$$V(z) = \frac{4\pi\rho}{4\pi\varepsilon_0 z}\left\{\int_0^z r'^2 dr' + \int_z^R z r' dr'\right\}$$

$$= \frac{4\pi\rho}{4\pi\varepsilon_0 z}\left\{\frac{z^3}{3} + \frac{zR^2}{2} - \frac{z^3}{2}\right\} = \frac{4\pi\rho}{8\pi\varepsilon_0}\left\{R^2 - \frac{z^2}{3}\right\}$$

$4\pi R^3\rho/3 = Q$

$$= \frac{Q}{8\pi\varepsilon_0 R}\left\{3 - \frac{z^2}{R^2}\right\}$$

Uniformly charged sphere (cont'd)

- *V*-->**E**

Outside $V(r) = \frac{Q}{4\pi\varepsilon_0 r}$ $\quad \mathbf{E}(r) = -\nabla V(r) = -\frac{Q}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\left[\frac{1}{r}\right]\hat{\mathbf{r}}$ $= \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$

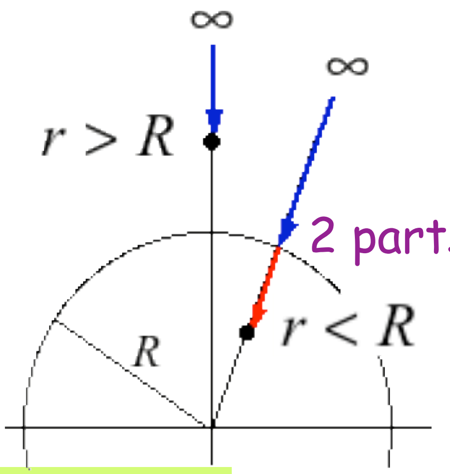
Inside $V(r) = \frac{Q}{8\pi\varepsilon_0 R}\left\{3 - \frac{r^2}{R^2}\right\}$ $\quad \mathbf{E}(r) = -\nabla V(r)$ $= \frac{Q}{4\pi\varepsilon_0}\frac{r}{R^3}\hat{\mathbf{r}}$

- **E**-->*V*

Outside $V(r) = -\int_\infty^r \mathbf{E}\cdot d\ell = -\frac{Q}{4\pi\varepsilon_0}\int_\infty^r \frac{dr'}{r'^2}$ $= \frac{Q}{4\pi\varepsilon_0 r}$

Inside $V(r) = -\frac{Q}{4\pi\varepsilon_0}\left\{\int_\infty^R \frac{dr'}{r'^2} + \int_R^r \frac{r'dr'}{R^3}\right\}$

$$= \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{R} - \frac{r^2}{2R^3} + \frac{1}{2R}\right\} = \frac{Q}{8\pi\varepsilon_0 R}\left\{3 - \frac{z^2}{R^2}\right\}$$



Potential energy for point charges

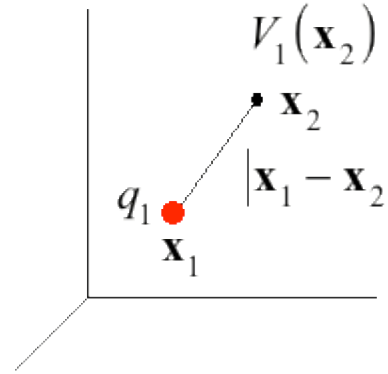
- Electric potential (should be a – sign in front of integral)

$$V(\mathbf{x}) = -\int_{\infty}^{\mathbf{x}} \mathbf{E} \cdot d\ell = W$$

Work done (by an external agent) moving a unit charge from ∞ to $\mathbf{x}$

$$V_1(\mathbf{x}_2) = \frac{q_1}{4\pi\varepsilon_0 \left|\mathbf{x}_1 - \mathbf{x}_2\right|}$$

Potential at $\mathbf{x}_2$ due to charge q_1 at point $\mathbf{x}_1$



- Potential energy

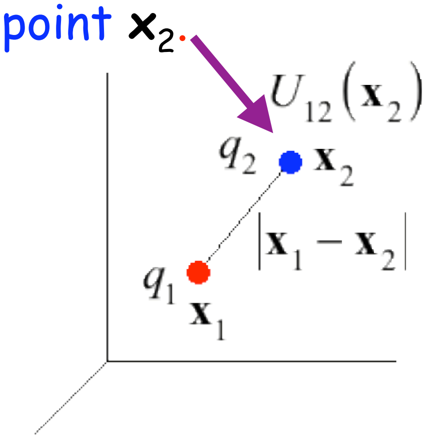
$$U(\mathbf{x}) = qV(\mathbf{x})$$

Potential energy U of charge q .

$$U_{12}(\mathbf{x}_2) = q_2 V_1(\mathbf{x}_2)$$

Potential energy U_{12} of the charge q_2 due the potential V_1

Put charge q_2 at point $\mathbf{x}_2$.



Potential energy for charge distributions

- Potential energy for multiple point charges

$$U = \frac{1}{2}\sum_{i\neq j} q_i V_j(\mathbf{x}_i, \mathbf{x}_j) = \frac{1}{2}\sum_{i\neq j} \frac{q_i q_j}{4\pi\varepsilon_0 \left|\mathbf{x}_i - \mathbf{x}_j\right|}$$

i ≠ j, do not count charge with itself. 1/2 corrects for $q_i\, q_j$ and $q_j\, q_i$

- Generalize for charge distributions

$$U = \frac{1}{2}\int \rho(\mathbf{x})d^3x \int \frac{\rho(\mathbf{x}')}{4\pi\varepsilon_0 |\mathbf{x}-\mathbf{x}'|}d^3x'$$

$$V(\mathbf{x}) = \int \frac{\rho(\mathbf{x}')}{4\pi\varepsilon_0 |\mathbf{x}-\mathbf{x}'|}d^3x'$$

$$U = \frac{1}{2}\int \rho(\mathbf{x})V(\mathbf{x})d^3x$$

Charge produces potential and the same charge gains potential energy?

- Where is this energy?

The charge distribution produces an electric field. We interpret the energy as being stored by (in) the field !

Energy of an electric field

- Express U as a function of $\mathbf{E}$ only !

$$U = \frac{1}{2}\int \rho V d^3x$$

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$$

Identity

$$U = \frac{\varepsilon_0}{2}\int (\nabla \cdot \mathbf{E}) V d^3x$$

$$\nabla \cdot (V\mathbf{E}) = V\nabla \cdot \mathbf{E} + \mathbf{E} \cdot (\nabla V)$$

$$= \frac{\varepsilon_0}{2}\int \nabla \cdot (V\mathbf{E}) d^3x - \frac{\varepsilon_0}{2}\int \mathbf{E} \cdot (\nabla V) d^3x$$

$$\mathbf{E} = -\nabla V$$

Gauss's theorem

$$= \frac{\varepsilon_0}{2}\int \nabla \cdot (V\mathbf{E}) d^3x + \frac{\varepsilon_0}{2}\int E^2 d^3x$$

$$\oint_{Vol} \nabla \cdot (V\mathbf{E}) d^3x = \oint_S V\mathbf{E} \cdot d\mathbf{A}$$

$$= \frac{\varepsilon_0}{2}\oint_S V\mathbf{E} \cdot d\mathbf{A} + \frac{\varepsilon_0}{2}\oint_{Vol} E^2 d^3x \xrightarrow[S,Vol\to\infty]{} U = \frac{\varepsilon_0}{2}\oint_{all} E^2 d^3x$$

Field energy

Energy density $u_E(\mathbf{x}) = \varepsilon_0 E^2(\mathbf{x})/2$