PHY481: Electromagnetism

Multipoles

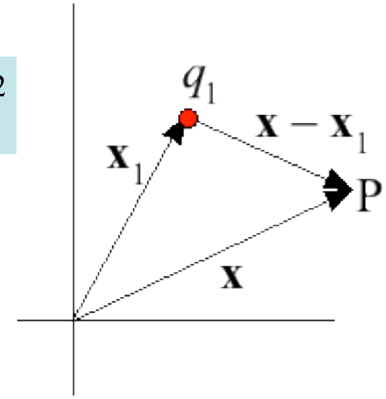
HW 3b hints

Approximating 1/|x-x′|

Potential of single charge

$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{|\mathbf{x}-\mathbf{x}_1|}$$

$$|\mathbf{x}| = r; \quad |\mathbf{x}_1| = r_1; \quad |\mathbf{x}-\mathbf{x}_1|^2 = r^2 - 2rr_1\cos\theta_1 + r_1^2$$



$$\frac{1}{|\mathbf{x}-\mathbf{x}_1|} = \frac{1}{\sqrt{r^2 - 2rr_1\cos\theta_1 + r_1^2}} = \frac{1}{r\sqrt{1 - 2\frac{r_1}{r}\cos\theta_1 + \frac{r_1^2}{r^2}}} = \frac{1}{r\sqrt{1-\varepsilon}}$$

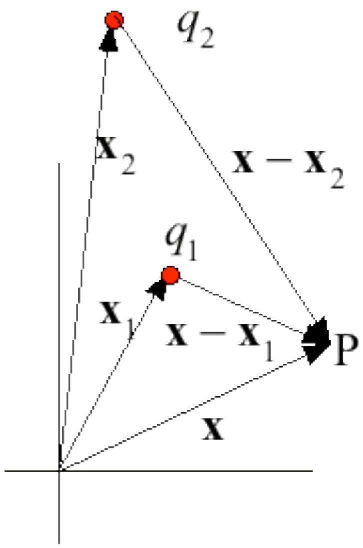
$$\frac{1}{\sqrt{1-\varepsilon}} = \left(1 + \frac{\varepsilon}{2} + \frac{3\varepsilon^2}{8} + \ldots\right); \quad \varepsilon = 2\frac{r_1}{r}\cos\theta_1 - \frac{r_1^2}{r^2}; \quad \varepsilon^2 = 4\frac{r_1^2}{r^2}\cos^2\theta_1 - 4\frac{r_1^3}{r^3}\cos\theta_1 + \frac{r_1^4}{r^4}$$

$$\frac{1}{|\mathbf{x}-\mathbf{x}_1|} == \frac{1}{r}\left(1 + \frac{\varepsilon}{2} + \frac{3\varepsilon^2}{8} + \ldots\right) = \frac{1}{r}\left(1 + \frac{r_1}{r}\cos\theta_1 - \frac{r_1^2}{2r^2} + \frac{3}{8}\left(4\frac{r_1^2}{r^2}\cos^2\theta_1 - 4\frac{r_1^3}{r^3}\cos\theta_1 + \frac{r_1^4}{r^4}\right) + \ldots\right)$$

Approximating 1/|x-x'|

Potential of two charges

$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{|\mathbf{x}-\mathbf{x}_1|} + \frac{q_2}{|\mathbf{x}-\mathbf{x}_2|}\right)$$



$$\frac{q_1}{|\mathbf{x}-\mathbf{x}_1|} + \frac{q_2}{|\mathbf{x}-\mathbf{x}_2|} = \frac{Q}{r} + \frac{q_1 r_1 \cos\theta_1 + q_2 r_2 \cos\theta_2}{r^2} + \ldots$$

$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{r^2} + \frac{\hat{\mathbf{r}}\cdot\mathbf{Q_2}\cdot\hat{\mathbf{r}}}{r^3} + \ldots\right)$$

dipole moment

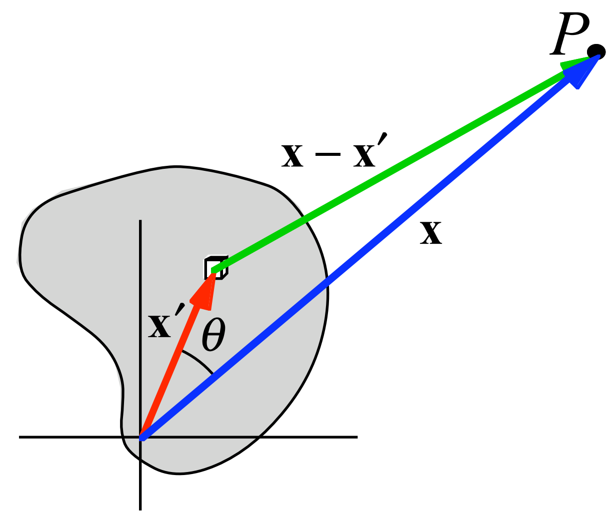
$$\mathbf{p} = q_1\mathbf{x}_1 + q_2\mathbf{x}_2$$

quadrupole moment

$$\mathbf{Q_2} = \frac{q_1}{2}\left(3\mathbf{x}_1\mathbf{x}_1 - r_1^2\mathbf{I}\right) + \frac{q_2}{2}\left(3\mathbf{x}_2\mathbf{x}_2 - r_2^2\mathbf{I}\right)$$

Multipole expansion of V

$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$



monopole term

$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{x}')d^3x' + \right.$$

dipole term

$$+\frac{1}{r^2}\int r'\cos\theta\,\rho(\mathbf{x}')d^3x'$$

quadrupole term

$$+\frac{1}{r^3}\int r'^2\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right)\rho(\mathbf{x}')d^3x'$$

higher-pole terms

$$\left. + \ldots \right]$$

$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0 r}\sum_{n=0}^{\infty}\int (r'/r)^n P_n(\cos\theta)\rho(\mathbf{x}')d^3x'$$

Legendre polynomials

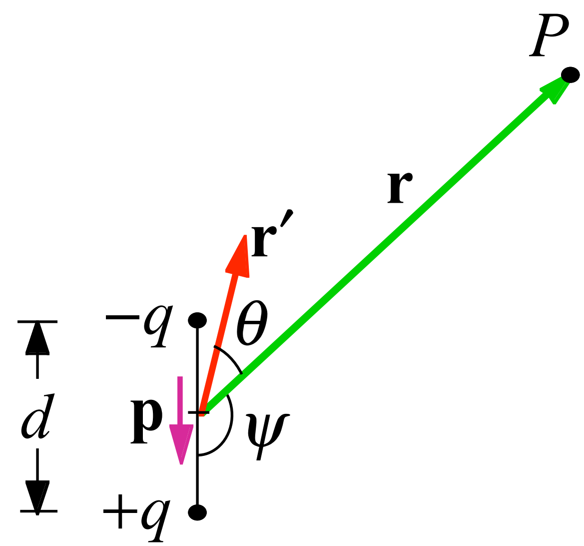
$$\underline{P_0(\cos\theta)} = 1;\ \underline{P_1(\cos\theta)} = \cos\theta;\ \underline{P_2(\cos\theta)} = \tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}$$

Potential of a finite dipole

- Charges $+q$ and $-q$ separated by d along the z-axis

Delta function charge density

$$\rho(\mathbf{r}') = \left[q\delta\left(\mathbf{r}' + \hat{\mathbf{k}}d/2\right) - q\delta\left(\mathbf{r}' - \hat{\mathbf{k}}d/2\right)\right]$$



monopole term

$$\int \rho(\mathbf{r}')\, d^3x' = 0$$

dipole term

$$r'\cos\theta = \hat{\mathbf{r}}\cdot\mathbf{r}'$$

$$\int r'\cos\theta\, \rho(\mathbf{r}')\, d^3x' = \int \hat{\mathbf{r}}\cdot\mathbf{r}'\rho(\mathbf{r}')\, d^3x' = \hat{\mathbf{r}}\cdot\int \mathbf{r}'\rho(\mathbf{r}')\, d^3x' = \hat{\mathbf{r}}\cdot\left(-qd\,\hat{\mathbf{k}}\right) = \hat{\mathbf{r}}\cdot\mathbf{p}$$

Angle ψ is between **p** and **r**

$$V(\mathbf{x}) \approx \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}$$ with $\mathbf{p} = -p\hat{\mathbf{k}},\ p = qd$ dipole moment

$$V(\mathbf{x}) \approx \frac{p\cos\psi}{4\pi\varepsilon_0 r^2}$$

Higher poles --> 0 ~ r^{-4} or faster

Point dipole

- Potential of a point dipole **p** (units are C•m)

Limit of finite dipole as $d \to 0$, with qd constant

$$p = \lim_{d\to 0}(qd)\Big|_{qd\ \text{constant}}$$

Exact

$$V(\mathbf{x}) = \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}$$

with $\mathbf{p} = p\hat{\mathbf{k}}$

$$V(r,\theta) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

- Electric field of point dipole $\mathbf{E}(\mathbf{x}) = -\nabla V(\mathbf{x})$

$$\mathbf{E}(\mathbf{x}) = \frac{-1}{4\pi\varepsilon_0}\nabla\left(\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}\right) = \frac{-1}{4\pi\varepsilon_0}\nabla\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right)$$

$$= \frac{-1}{4\pi\varepsilon_0}\left[\mathbf{p}\cdot\mathbf{r}\nabla\left(\frac{1}{r^3}\right) + \left(\frac{1}{r^3}\right)\nabla(\mathbf{p}\cdot\mathbf{r})\right]$$

$$\nabla(\mathbf{p}\cdot\mathbf{r}) = \hat{\mathbf{e}}_i\frac{\partial}{\partial x_i}(p_j x_j) = \hat{\mathbf{e}}_i p_i = \mathbf{p}$$

$$= \frac{-1}{4\pi\varepsilon_0}\left[\mathbf{p}\cdot\mathbf{r}\left(\frac{-3}{r^4}\hat{\mathbf{r}}\right) + \left(\frac{\mathbf{p}}{r^3}\right)\right] = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right]$$

Torque and energy of dipole in electric field

- Torque on electric dipole

$$\mathbf{N} = \mathbf{p} \times \mathbf{E} = pE\sin\theta(-\hat{\mathbf{k}})$$

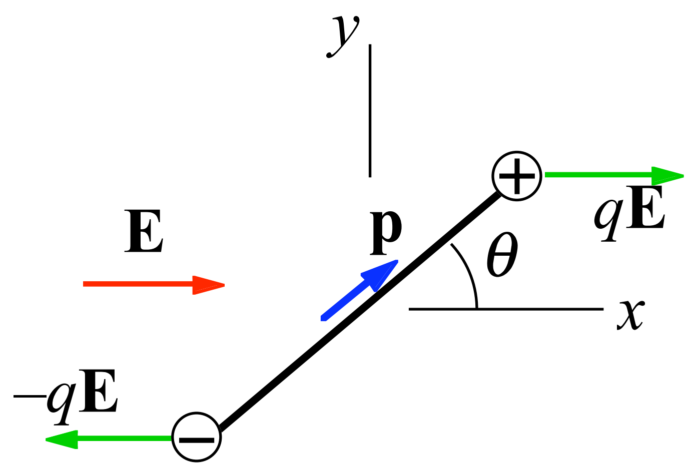
- Energy of electric dipole

$$U = -\mathbf{p} \cdot \mathbf{E} = -pE\cos\theta$$

- Force on electric dipole

$$\mathbf{F}(\mathbf{x}) = -\nabla U(\mathbf{x}) = \nabla\left(\mathbf{p} \cdot \mathbf{E}(\mathbf{x})\right) = \left(\mathbf{p} \cdot \nabla\right)\mathbf{E}(\mathbf{x})$$

$$F_j(\mathbf{x}) = p_i \frac{\partial E_j(\mathbf{x})}{\partial x_i}$$



If $\mathbf{E}(\mathbf{x})$ is constant, $\mathbf{F}(\mathbf{x}) = 0$

Problem 2.26

Self energy of a sphere with charge Q and radius R.

Two ways to get the energy:

1) Build up charge

Do the work necessary
to assemble the charge

2) Surface integral

Self energy calculation

Both should give the same answer, that diverges as R -> 0

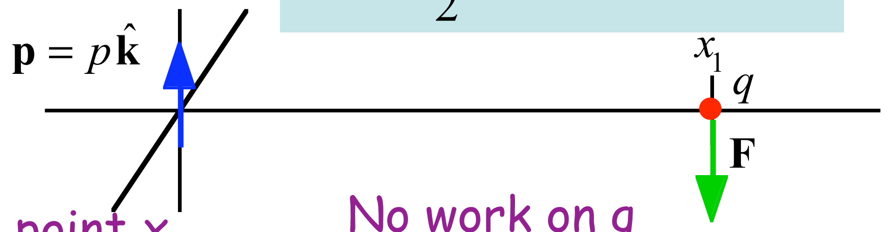
Problem 3.32

a) Point dipole force on a displaced charge

$$\text{at } \theta = \frac{\pi}{2},\ \cos\theta = 0, \hat{\boldsymbol{\theta}} = -\hat{\mathbf{k}}$$

Field of dipole **p** (in +z direction)

$$\mathbf{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$



Field at point x_1

$$\mathbf{E} = \frac{-p\hat{\mathbf{k}}}{4\pi\varepsilon_0 x_1^3}$$

Force on q at point x_1

$$\mathbf{F} = q\mathbf{E}$$

No work on q

$$W$$

F on dipole = -**F** on charge

Check this by direct calculation

Consideer a bar between the point charge and the dipole

Will it rotate when released?