
PHY481: Electromagnetism

HW 3b Solutions
Practice Exam Hints

HW 3 Problem 3.22

Potential inside and outside a spherical shell, uniform charge density
(See lecture 11)

$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}') d^3x'}{|\mathbf{x} - \mathbf{x}'|} \quad 4\pi R^2 \sigma = Q$$

$$V(z) = \frac{\sigma R^2}{4\pi\epsilon_0} \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \left[z^2 + R^2 - 2zR \cos\theta \right]^{-1/2}$$

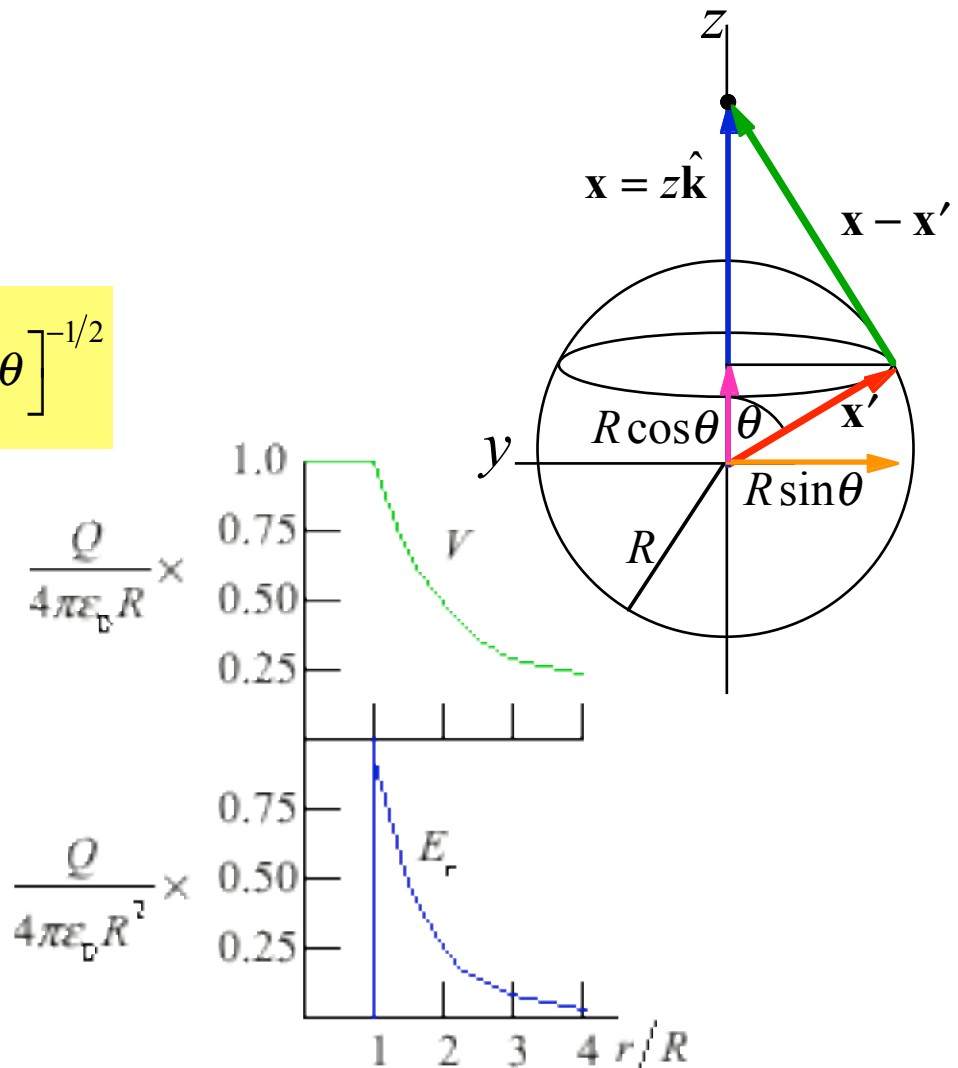
$$V(z) = \frac{2\pi\sigma R}{4\pi\epsilon_0 z} [(R+z) - (\pm)(R-z)] \begin{matrix} (-) & z > R \\ (+) & z < R \end{matrix}$$

$$V(z) = \frac{Q}{4\pi\epsilon_0 R}, \quad z < R$$

$$V(z) = \frac{Q}{4\pi\epsilon_0 z}, \quad z > R$$

$$\mathbf{E}(z) = -\nabla V = 0, \quad r < R$$

$$E_r(r) = \frac{Q}{4\pi\epsilon_0 r^2}, \quad r > R$$



HW 3 Problem 3.23

a) Potential and field inside and outside a sphere, charge density $\rho = Cr$
 (See lecture 14 for one technique, or **use Gauss's law for E**, then $E \rightarrow V$)

Gauss's law $\oint_S \mathbf{E} \cdot d\mathbf{A} = q/\epsilon_0$

$$Q = \int \rho(\mathbf{x}') d^3x' = 4\pi \int_0^a Cr'^3 dr' = \pi Ca^4; \quad C = \frac{Q}{\pi a^4}$$

Inside sphere ($r < a$)

Outside sphere ($r > a$)

$$4\pi r^2 E = \frac{4\pi}{\epsilon_0} \int_0^r Cr'^3 dr' = \frac{\pi}{\epsilon_0} Cr^4 \quad E_r = \underline{Cr^2/4\epsilon_0}$$

$$4\pi r^2 E = Q/\epsilon_0 = \pi Ca^4/\epsilon_0 \quad E_r = \underline{Ca^4/4\epsilon_0 r^2}$$

$E \rightarrow V$

$$V(r) = -\int_{\infty}^r \mathbf{E} \cdot d\ell$$

Inside sphere

Outside sphere

$$V(r) = -\frac{Ca^4}{4\epsilon_0} \int_{\infty}^a r'^{-2} dr' - \frac{C}{4\epsilon_0} \int_a^r r'^2 dr' = \frac{C}{12\epsilon_0} [4a^3 - r^3]$$

$$V(r) = -\frac{Ca^4}{4\epsilon_0} \int_{\infty}^r r'^{-2} dr' = \underline{\frac{Ca^4}{4\epsilon_0 r}}$$

b) Add a surface charge density σ_0 to the sphere

Inside sphere

Outside sphere

$$V(r) = \underline{\frac{\sigma_0 R}{\epsilon_0} + \frac{C}{12\epsilon_0} [4a^3 - r^3]} \quad E_r = \underline{\frac{Cr^2}{4\epsilon_0}}$$

$$V(r) = \underline{\frac{\sigma_0 a^2}{\epsilon_0 r} + \frac{Ca^4}{4\epsilon_0 r}} \quad E_r = \underline{\frac{\sigma_0 a^2}{\epsilon_0 r^2} + \frac{Ca^4}{4\epsilon_0 r^2}}$$

HW 3 Problem 3.25

Laplace's equation in a charge free region

Poisson's equation $-\nabla^2 V(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\epsilon_0}$

In charge free regions
Laplace's equation

$$\nabla^2 V(\mathbf{x}) = 0$$

$$V(r, \theta) = r^{-3} (C_1 \cos^2 \theta + C_2)$$

$$\frac{\partial V}{\partial r} = -3r^{-4} (C_1 \cos^2 \theta + C_2)$$

$$\nabla^2 V(r, \theta) = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial V}{\partial r} \right] + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \frac{\partial V}{\partial \theta} \right]$$

$$\frac{\partial V}{\partial \theta} = -2r^{-3} C_1 \cos \theta \sin \theta$$

$$\nabla^2 V(r, \theta) = 0 = \frac{1}{r^2} \frac{\partial}{\partial r} \left[-3r^{-2} (C_1 \cos^2 \theta + C_2) \right] + \frac{-2C_1}{r^5 \sin \theta} \frac{\partial}{\partial \theta} [\cos \theta \sin^2 \theta]$$

$$0 = \frac{6}{r^5} (C_1 \cos^2 \theta + C_2) - \frac{2C_1}{r^5} [-1 + 3\cos^2 \theta]$$

$$C_1 = -3C_2$$

HW 3 Problem 3.26

Self energy of a sphere with charge Q and radius R .

Two ways to get the energy:

1) Build up charge

$$U = \int dU = \int V dq$$

Do the work necessary
to assemble the charge

$$U = \int V dq = \frac{1}{4\pi\epsilon_0 R} \int_0^Q q dq = \frac{Q^2}{8\pi\epsilon_0 R}$$

2) Surface integral

$$U = \frac{1}{2} \int \rho(\mathbf{x}) V(\mathbf{x}) d^3x'$$

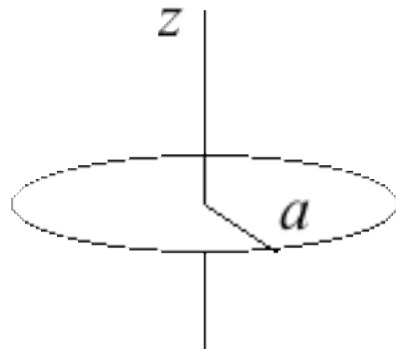
Self energy
calculation

$$U = \frac{1}{2} Q \frac{Q}{4\pi\epsilon_0 R} = \frac{Q^2}{8\pi\epsilon_0 R}$$

Both give the same answer, that diverges as $R \rightarrow 0$

HW 3 Problem 3.30

Multipole expansion for the potential on z-axis for a ring of charge.



Taylor series expansion of $f(x)$ about $x = 0$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \dots$$

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2$$

$$\begin{aligned} V(0,0,z) &= \frac{1}{4\pi\epsilon_0} \frac{Q}{(a^2 + z^2)^{1/2}} = \frac{Q}{4\pi\epsilon_0 z} \left(1 + \frac{a^2}{z^2} \right)^{-1/2} \\ &= \frac{Q}{4\pi\epsilon_0 z} \left(1 - \frac{a^2}{2z^2} + \frac{3a^4}{8z^4} + \dots \right) \end{aligned}$$

monopole

quadrupole

hexadecapole

No dipole term as there is a complete symmetry in z for the charge distribution

HW 3 Problem 3.32

a) Point dipole force on a displaced charge

Field of dipole $\mathbf{p}$ (in +z direction, vs. r & θ)

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3} (2\cos\theta \hat{\mathbf{r}} + \sin\theta \hat{\boldsymbol{\theta}})$$

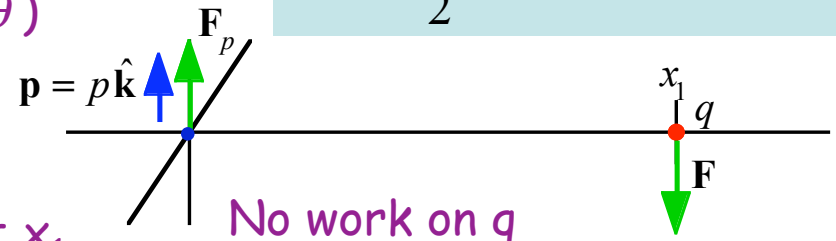
Field at point x_1

$$\mathbf{E} = \frac{-p \hat{\mathbf{k}}}{4\pi\epsilon_0 x_1^3}$$

Force on q at point x_1

$$\mathbf{F} = q\mathbf{E} = \frac{-qp \hat{\mathbf{k}}}{4\pi\epsilon_0 r^3}$$

$$\text{at } \theta = \frac{\pi}{2}, \cos\theta = 0, \hat{\boldsymbol{\theta}} = -\hat{\mathbf{k}}$$



$$W = \int_{\infty}^{x_1} \mathbf{F} \cdot \hat{\mathbf{i}} dx = 0$$

Newton's 3rd law

$\mathbf{F}$ on dipole = $-\mathbf{F}$ on charge

$$\mathbf{F}_p = \frac{+qp \hat{\mathbf{k}}}{4\pi\epsilon_0 x_1^3}$$

Check by direct calculation

$$F_z = (\mathbf{p} \cdot \nabla) E_z = p_z \frac{\partial}{\partial z} E_z = \frac{qp}{4\pi\epsilon_0} \frac{\partial}{\partial z} \left(\frac{z}{[(x-x_1)^2 + y^2 + z^2]^{3/2}} \right)$$

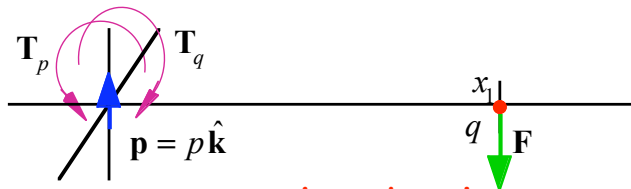
$$F_z|_{x,y,z=0} = \frac{qp}{4\pi\epsilon_0} \left(\frac{1}{r^3} - \frac{3z^2}{r^5} \right) \bigg|_{x,y,z=0} = \frac{qp}{4\pi\epsilon_0 x_1^3}$$

Forces are "balanced", but what about torque?

Will a bar between the point charge and the dipole rotate if released?

Violates angular momentum conservation: No external torques $\rightarrow \mathbf{L}$ is constant

HW 3 Problem 3.32 (cont'd)



they balance

Torque from the charge

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \hat{\mathbf{j}} qp / 4\pi\epsilon_0 x_1^2$$

Use pivot at the dipole

Torque from the dipole

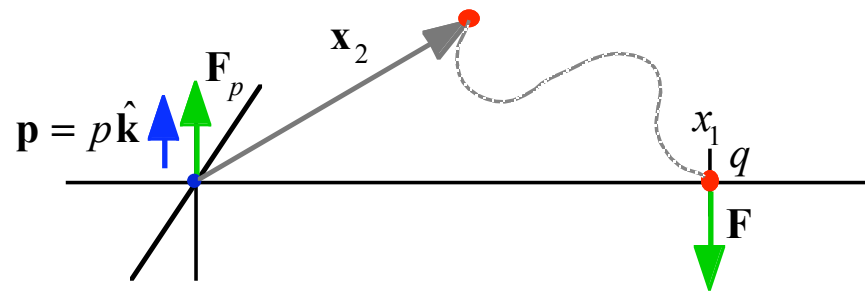
$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} = -\hat{\mathbf{j}} qp / 4\pi\epsilon_0 x_1^2$$

b) Force on q at point $\mathbf{x}_2$, and work needed to move charge there.

Work needed = Potential energy change

$$\Delta U = qV(\mathbf{x}_2) - qV(\mathbf{x}_1); \quad V(\mathbf{x}_1) = 0$$

$$U(\mathbf{x}_2) = q \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi\epsilon_0 r^3} = q \frac{pz_2}{4\pi\epsilon_0 r_2^3}$$



$\mathbf{F} = q \mathbf{E}$ at the point $\mathbf{x}_2$

$$\mathbf{F} = q\mathbf{E} = \frac{qp}{4\pi\epsilon_0 r_2^3} (2\cos\theta_2 \hat{\mathbf{r}} + \sin\theta_2 \hat{\boldsymbol{\theta}}) = \frac{qp}{4\pi\epsilon_0 r_2^4} (2z_2 \hat{\mathbf{r}} + (x_2^2 + y_2^2)^{1/2} \hat{\boldsymbol{\theta}})$$

$$\mathbf{F} = \frac{qp}{4\pi\epsilon_0 r_2^3} (3\cos\theta_2 \hat{\mathbf{r}} - \hat{\mathbf{k}}) = \frac{qp}{4\pi\epsilon_0 r_2^3} \left(3 \frac{z_2}{r_2} \hat{\mathbf{r}} - \hat{\mathbf{k}} \right) \quad \text{using} \quad \hat{\mathbf{k}} = \cos\theta_2 \hat{\mathbf{r}} - \sin\theta_2 \hat{\boldsymbol{\theta}}$$

HW 3 Problem 3.33

Dipole field and potential in spherical coordinates

$$\mathbf{p} = p_1 \hat{\mathbf{i}} + p_2 \hat{\mathbf{j}} + p_3 \hat{\mathbf{k}}$$

$$\mathbf{r} = x \hat{\mathbf{i}} + y \hat{\mathbf{j}} + z \hat{\mathbf{k}}$$

$$V = \frac{\mathbf{p} \cdot \mathbf{r}}{4\pi\epsilon_0 r^3} = \frac{1}{4\pi\epsilon_0 r^2} \left(p_1 \frac{x}{r} + p_2 \frac{y}{r} + p_3 \frac{z}{r} \right)$$

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$V = \frac{1}{4\pi\epsilon_0 r^2} (p_1 \sin \theta \cos \phi + p_2 \sin \theta \sin \phi + p_3 \cos \theta)$$

$$\mathbf{E} = -\nabla V = -\left\{ \frac{\partial V}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \hat{\boldsymbol{\phi}} \right\}$$

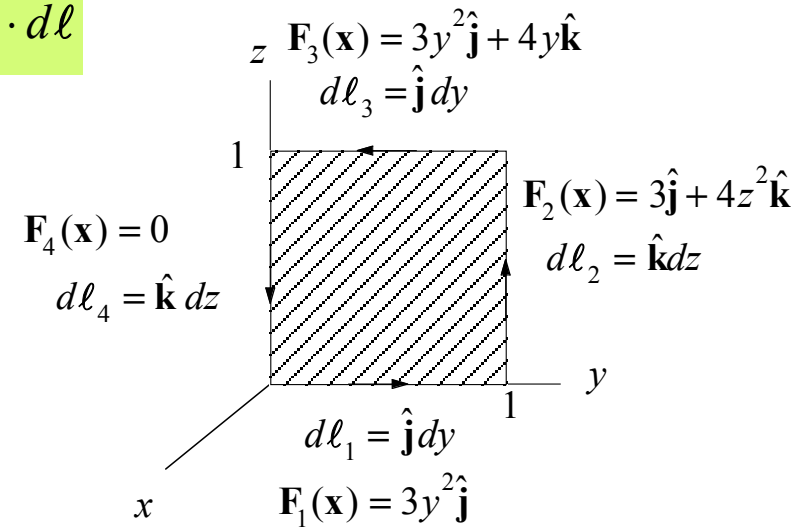
$$\begin{aligned} \mathbf{E} = \frac{1}{4\pi\epsilon_0 r^3} \{ & 2(p_1 \sin \theta \cos \phi + p_2 \sin \theta \sin \phi + p_3 \cos \theta) \hat{\mathbf{r}} \\ & + (-p_1 \cos \theta \cos \phi - p_2 \cos \theta \sin \phi + p_3 \sin \theta) \hat{\boldsymbol{\theta}} + \\ & + (p_1 \sin \phi - p_2 \cos \phi) \hat{\boldsymbol{\phi}} \} \end{aligned}$$

Practice problems

1) Stokes's theorem: $\oint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} dA = \oint_C \mathbf{F} \cdot d\boldsymbol{\ell}$

$$\mathbf{F}(\mathbf{x}) = (2xy + 3y^2)\hat{\mathbf{j}} + (4yz^2)\hat{\mathbf{k}}$$

$$\oint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = \oint_C \mathbf{F} \cdot d\boldsymbol{\ell} = \underline{4/3}$$



2) Gauss's law

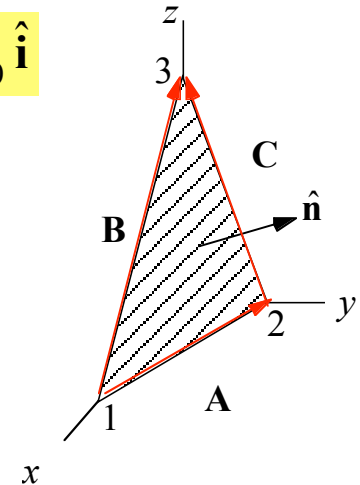
a) Cross product of 2 vectors in the plane $\mathbf{A} = y_0\hat{\mathbf{j}} - x_0\hat{\mathbf{i}}$

$$\hat{\mathbf{n}} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}$$

b) Dot product of normal with vector in the plane

$$\hat{\mathbf{n}} \cdot (\mathbf{x} - \mathbf{x}') = 0$$

$$\mathbf{x} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$$



Practice problems

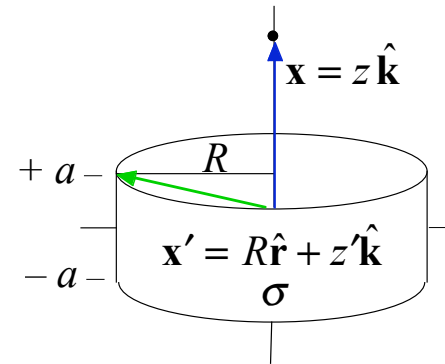
3a) Gauss's law: $\oint_S \mathbf{E} \cdot d\mathbf{A} = q_{encl} / \epsilon_0$

$$E_{plane} = \frac{\sigma}{2\epsilon_0}$$

4) Electric field for ring of charge

$$E = \frac{1}{4\pi\epsilon_0} \int \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} \rho(\mathbf{x}') d^3x'$$

$$E_z = \frac{\sigma}{4\pi\epsilon_0} \int_0^{2\pi} R d\phi' \int_{-a}^{+a} dz' \frac{(z - z')}{(R^2 + (z - z')^2)^{3/2}}$$



b) Approximation for $z \gg a, R$ Write as function of $\frac{a}{z}; \frac{R}{z}$

5) Prove $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$C_k = \epsilon_{klm} A_l B_m$$

Practice Problems

6) Point dipole potential $\mathbf{p} = p_0 \hat{\mathbf{k}}$ $V(r, \theta) = \frac{p_0 \cos \theta}{4\pi\epsilon_0 r^2}$

$$\mathbf{E} = -\nabla V = -\left(\hat{\mathbf{r}} \frac{\partial V}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \right)$$

7) Dirac delta function $\int d^3x' f(\mathbf{x}') \delta^3(\mathbf{x} - \mathbf{x}') = f(\mathbf{x})$

a) (i) $\int_2^6 (3x^2 - 2x - 1) \delta(x - 3) = (3 \cdot 9 - 2 \cdot 3 - 1)$

Practice Problems

7) Dirac delta function $\int d^3x' f(\mathbf{x}') \delta^3(\mathbf{x} - \mathbf{x}') = f(\mathbf{x})$

a) (i) $\int_2^6 (3x^2 - 2x - 1) \delta(x - 3) = (3 \cdot 9 - 2 \cdot 3 - 1)$