PHY481: Electromagnetism

Exam 1 Solutions

Charge on a conductor

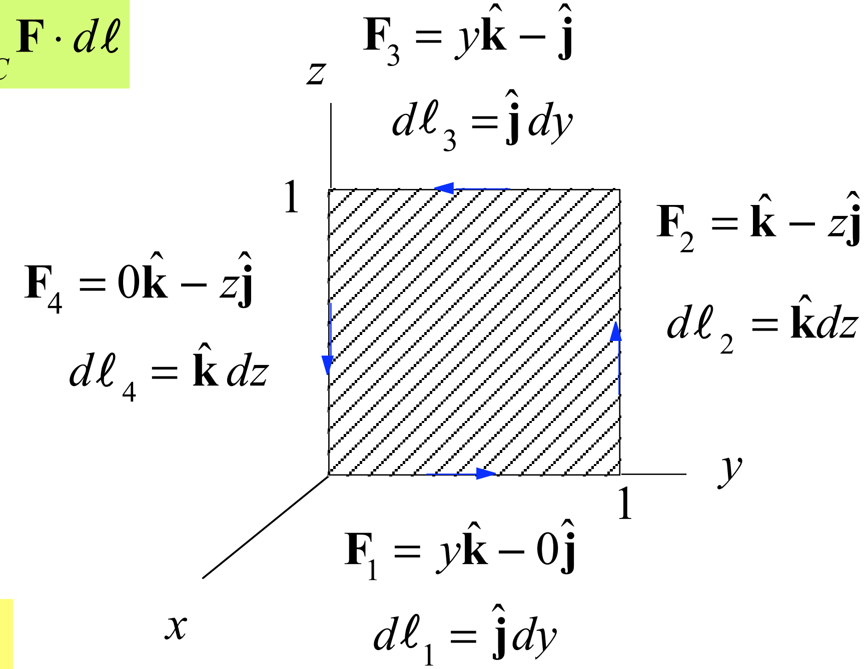
Exam 1

1) Stokes's theorem: $\oint_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dA = \oint_C \mathbf{F}\cdot d\boldsymbol{\ell}$

$$\mathbf{F}(\mathbf{x}) = \hat{\mathbf{i}}\times(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}})$$
$$= y\hat{\mathbf{k}} - z\hat{\mathbf{j}}$$

1) $d\boldsymbol{\ell} = \hat{\mathbf{j}}dy,\ z=0,\quad$ 2) $d\boldsymbol{\ell} = \hat{\mathbf{k}}dz,\ y=1$

3) $d\boldsymbol{\ell} = \hat{\mathbf{j}}dy,\ z=1,\quad$ 4) $d\boldsymbol{\ell} = \hat{\mathbf{k}}dz,\ y=0$



a) $\int_1 \mathbf{F}\cdot d\boldsymbol{\ell} = -0\int_0^1 dy = \underline{0};\quad \int_2 \mathbf{F}\cdot d\boldsymbol{\ell} = 1\int_0^1 dz = \underline{1};$

$\int_3 \mathbf{F}\cdot d\boldsymbol{\ell} = -1\int_1^0 dy = \underline{1};\quad \int_4 \mathbf{F}\cdot d\boldsymbol{\ell} = -0\int_1^0 dz = \underline{0}$

b) $\int_C \mathbf{F}\cdot d\boldsymbol{\ell} = \underline{2}$

$$\nabla\times\mathbf{F} = \hat{\mathbf{i}}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = \hat{\mathbf{i}}(2) + \hat{\mathbf{j}}(0) + \hat{\mathbf{k}}(0)$$

c) $\hat{\mathbf{n}} = \hat{\mathbf{i}};\quad dA = dxdy\quad;\quad \int_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dA = \underline{2}\quad \therefore\quad \int_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dA = \int_C \mathbf{F}\cdot d\boldsymbol{\ell} = \underline{2}$

Exam 1 (cont'd)

2)

a) integral to obtain scalar potential from charge distribution

$$V(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}$$

b) differential equation relating charge density and the electric field

$$\nabla \cdot \mathbf{E}(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\varepsilon_0}$$

c) differential equation relating electric field and potential

$$\mathbf{E}(\mathbf{x}) = -\nabla V(\mathbf{x})$$

d) integral to obtain electric field from the charge distribution

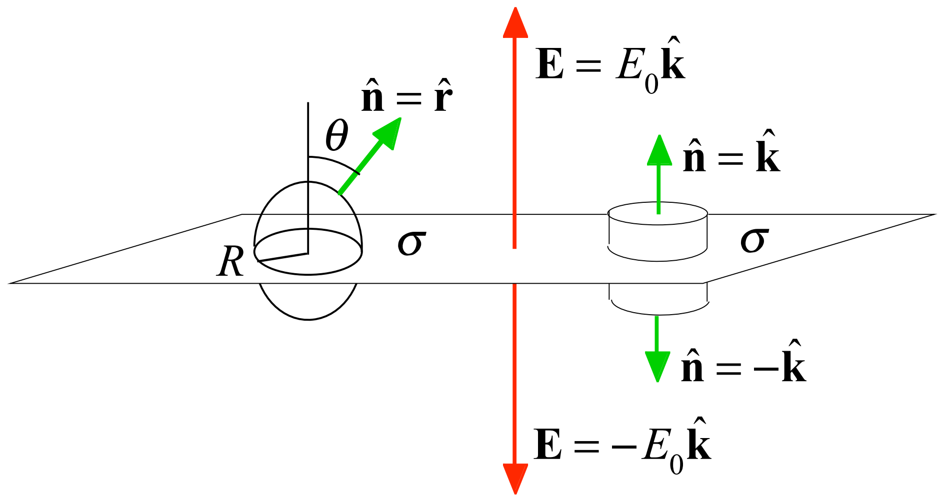
$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x'$$

Exam 1 (cont'd)

3a) Gauss's law: $\oint_S \mathbf{E}\cdot d\mathbf{A} = q_{encl}/\varepsilon_0$

b) $\mathbf{E}_T(\mathbf{x}) = E_0\hat{\mathbf{k}};\quad \mathbf{E}_B(\mathbf{x}) = -E_0\hat{\mathbf{k}}$

$\hat{\mathbf{n}}_T = \hat{\mathbf{k}};\quad \hat{\mathbf{n}}_B = -\hat{\mathbf{k}}$



$$\int_S \mathbf{E}\cdot d\mathbf{A} = \int_T E_0\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}\,dA + \int_B \left(-E_0\hat{\mathbf{k}}\right)\cdot\left(-\hat{\mathbf{k}}\,dA\right)$$

$$2E_0A = q_{encl}/\varepsilon_0 = \sigma A/\varepsilon_0$$

$$E_0 = \sigma/2\varepsilon_0$$

c) $\hat{\mathbf{n}} = \hat{\mathbf{r}};\quad dA = R^2\sin\theta d\theta d\phi;\quad q_{encl} = \sigma\pi R^2$

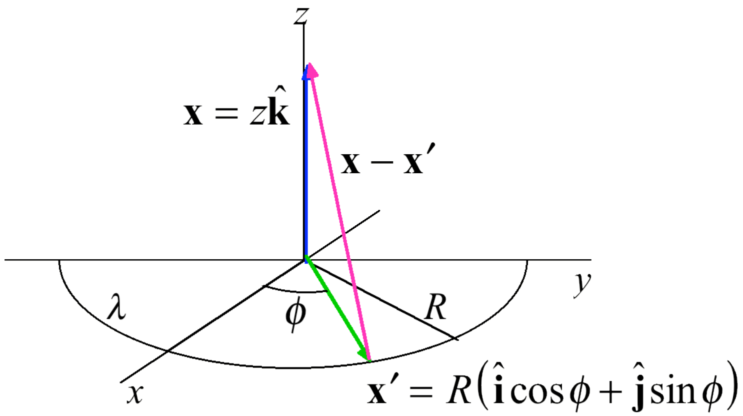
$$\int_S \mathbf{E}\cdot d\mathbf{A} = \int_T \left(E_0\hat{\mathbf{k}}\right)\cdot\hat{\mathbf{r}}\,dA + \int_B \left(-E_0\hat{\mathbf{k}}\right)\cdot\hat{\mathbf{r}}\,dA = 2\pi E_0R^2\left[\int_0^{\frac{\pi}{2}} d\theta\cos\theta\sin\theta - \int_{\frac{\pi}{2}}^{\pi} d\theta\cos\theta\sin\theta\right]$$

$$= 2\pi E_0R^2\left[\int_0^1 xdx - \int_1^0 xdx\right] = 2\pi E_0R^2\left[2\frac{x^2}{2}\Big|_0^1\right] = 2\pi E_0R^2$$

$$2\pi E_0R^2 = q_{encl}/\varepsilon_0 = \sigma\pi R^2/\varepsilon_0 \quad \therefore \quad E_0 = \sigma/2\varepsilon_0$$

Exam 1 (cont'd)

4a) $$q = \int \rho(x')d^3x' = \int \lambda d\ell = \int_{-\pi/2}^{\pi/2} Rd\phi = \pi R\lambda$$



b) $$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d^3x'$$

c) $$|\mathbf{x}-\mathbf{x}'|^3 = \left|z\hat{\mathbf{k}} - R\left(\hat{\mathbf{i}}\cos\phi + \hat{\mathbf{j}}\sin\phi\right)\right|^3 = \left(z^2+R^2\right)^{3/2}$$

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{z\hat{\mathbf{k}} - R\left(\hat{\mathbf{i}}\cos\phi + \hat{\mathbf{j}}\sin\phi\right)}{\left(z^2+R^2\right)^{3/2}}\lambda R d\phi$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{\lambda R}{\left(z^2+R^2\right)^{3/2}}\pi z\hat{\mathbf{k}} + \frac{\lambda R^2}{\left(z^2+R^2\right)^{3/2}}\int_{-\pi/2}^{\pi/2}\left(\hat{\mathbf{i}}\cos\phi + \hat{\mathbf{j}}\sin\phi\right)d\phi\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda R}{\left(z^2+R^2\right)^{3/2}}\left(\pi z\hat{\mathbf{k}} - 2R\hat{\mathbf{i}}\right)$$

d) $$E_z(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\frac{\pi\lambda Rz}{\left(z^2+R^2\right)^{3/2}} = \frac{q}{4\pi\varepsilon_0}\frac{z}{z^3\left(1+(R/z)^2\right)^{3/2}}$$

$R/z \to 0$ (i.e., $z >> R$)

$$E_z(\mathbf{x}) \approx \frac{q}{4\pi\varepsilon_0 z^2}$$

Exam 1 (cont'd)

5) Identity for $\nabla\cdot(\mathbf{A}\times\mathbf{B})$ using $\nabla\times\mathbf{A}$ and $\nabla\times\mathbf{B}$

$$\mathbf{C}=\mathbf{A}\times\mathbf{B} \text{ where } C_i=\varepsilon_{ijk}A_jB_k$$

$$\nabla\cdot(\mathbf{C})=\frac{\partial C_i}{\partial x_i}=\frac{\partial}{\partial x_i}\left(\varepsilon_{ijk}A_jB_k\right)=B_k\varepsilon_{ijk}\frac{\partial A_j}{\partial x_i}+A_j\varepsilon_{ijk}\frac{\partial B_k}{\partial x_i}$$

Make replacements $\varepsilon_{kij}=\varepsilon_{ijk}$ and $-\varepsilon_{jik}=\varepsilon_{ijk}$

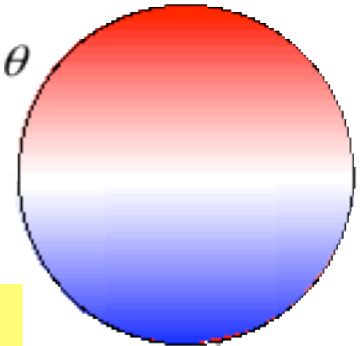
$$\begin{aligned}\nabla\cdot(\mathbf{C})&=B_k\left(\varepsilon_{kij}\right)\frac{\partial A_j}{\partial x_i}+A_j\left(-\varepsilon_{jik}\right)\frac{\partial B_k}{\partial x_i}\\&=B_k(\nabla\times\mathbf{A})_k-A_j(\nabla\times\mathbf{B})_j\end{aligned}$$

$$=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Exam 1 (cont'd)

6a) Dipole moment $\quad \mathbf{p} = \int \mathbf{r}\rho(\mathbf{x})d^3x$

$\sigma(\mathbf{x}) = \sigma_0 \cos\theta$



$$\hat{\mathbf{r}} = \hat{\mathbf{k}}\cos\theta + \sin\theta\left(\hat{\mathbf{i}}\cos\phi + \hat{\mathbf{j}}\sin\phi\right)$$

$$dA = R^2 \sin\theta d\theta d\phi$$

$$\mathbf{p} = 2\pi R^3 \int \hat{\mathbf{r}}\left(\sigma_0 \cos\theta\right)\sin\theta d\theta$$

$$= 2\pi R^3 \sigma_0 \left[\int_0^\pi \hat{\mathbf{k}}\cos^2\theta \sin\theta d\theta\right] + R^3\sigma_0 \int_0^\pi \cos\theta \sin^2\theta d\theta \int_0^{2\pi} d\phi\left(\hat{\mathbf{i}}\cos\phi + \hat{\mathbf{j}}\sin\phi\right)$$

$$= 2\pi R^3 \sigma_0 \hat{\mathbf{k}} \int_1^{-1} -x^2 dx = 2\pi R^3 \sigma_0 \hat{\mathbf{k}}\left[-\frac{x^3}{3}\right]_1^{-1}$$

$$= \frac{4\pi R^3 \sigma_0}{3}\hat{\mathbf{k}}$$

b) Top is positive, bottom is negative, the same as a dipole

c) From far away, potential is that of a point dipole : $V(r,\theta) = \dfrac{\mathbf{p}\cdot\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}$

Exam 1 (cont'd)

7a) Prove $\int_{-\infty}^{\infty} \delta(ax+b) f(x) dx = \frac{f(-b/a)}{|a|}$ (d^3x should be dx, answer OK)

$$x' = ax+b; \quad dx = dx'/|a|; \quad 2^{\text{nd}} \text{ integral is } f \text{ at } x'=0$$

$$\int_{-\infty}^{\infty} \delta(ax+b) f(x) dx = \frac{1}{a}\int_{-\infty}^{\infty} \delta(x') f(x'/a - b/a) dx' = \frac{f(-b/a)}{|a|}$$

b) Prove $\int_{-\infty}^{\infty} \delta^3(\mathbf{Ax}+\mathbf{b}) f(\mathbf{x}) d^3x = f(-\mathbf{A}^{-1}\mathbf{b}) / |\det \mathbf{A}|$ Jacobian $|\det \mathbf{A}|$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}' - \mathbf{A}^{-1}\mathbf{b}$$

$$d^3x = d^3x'/|\text{Det } \mathbf{A}|$$

$$\int_{-\infty}^{\infty} \delta(\mathbf{Ax}+\mathbf{b}) f(\mathbf{x}) d^3x = \int_{-\infty}^{\infty} \delta(\mathbf{x}') f(\mathbf{A}^{-1}\mathbf{x}' - \mathbf{A}^{-1}\mathbf{b}) d^3x'/|\text{Det } \mathbf{A}|$$

$$= f(-\mathbf{A}^{-1}\mathbf{b})/|\text{Det } \mathbf{A}|$$

Conductors in static equilibrium

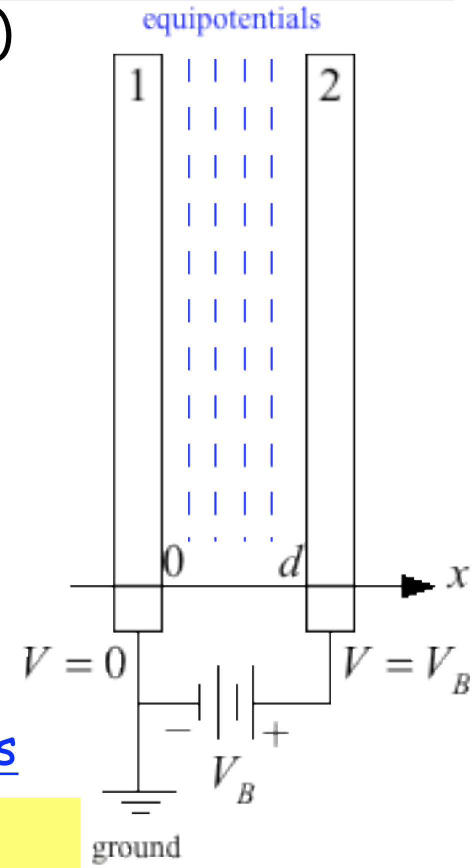
Why these statements are true ?

- Electric field **E** inside a conductor is zero.
- Potential V inside a conductor is a constant.
- Inside a conductor the charge density ρ is zero.
- Charge Q on a conductor resides only on the surface.
- A net charge Q_{net} here is always paired with charge $-Q_{net}$ elsewhere.
- On a conductor either V or Q_{net}, but not both, can be specified.
- At a conductor's surface, field component $E_t = 0$, component $E_n = \sigma/\varepsilon_0$, but in force calculations, $E_n = \sigma/2\varepsilon_0$ due to only "distant" charges.
- In an empty cavity in a conductor, **E** = 0 and surface $\sigma_{cavity} = 0$.
- The potential V of a conductor can be set by a battery V_B.
- The earth is an "infinite" source of charge at a constant V .
- V or Q_{net} of conductors (and $Q_{external}$) determine V everywhere.

Boundary value problems: 1-parallel plates

Parallel plates (find potential, field & surface charge)

The ground is a constant potential, so let $V_1 = 0$.
Battery draws electrons from plate 2 and puts them on plate 1. Process "continues" until $V_2 = V_B$.



Potential between the plates

Between the plates the potential depends only on *x*.

Laplace's equation: $$\nabla^2 V(x) = 0$$

Boundary conditions: $$V(0) = 0; \quad V(d) = V_B$$

General solution (ODE)

$$\nabla^2 V = \frac{d}{dx}\left(\frac{dV}{dx}\right) = 0; \; \frac{dV}{dx} = c_1$$

$$V(x) = \underline{c_1 x + c_2}$$

Apply boundary conditions

$$V(0) = \underline{c_2 = 0}$$

$$V(d) = c_1 d = V_B; \underline{c_1 = \frac{V_B}{d}}$$

Potential

$$V(x) = \underline{V_B \frac{x}{d}}$$

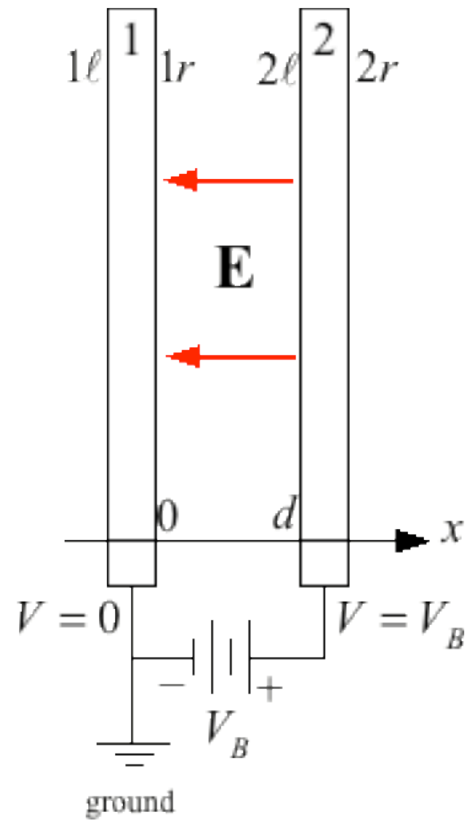
Parallel plates (cont'd)

Electric field

$$\mathbf{E} = -\nabla V = -\nabla\left(V_B \frac{x}{d}\right)$$

Electric field

$$-\frac{V_B}{d}\hat{\mathbf{i}}$$



E_n -> Surface charges

$$E_{1n} = \frac{\sigma_{1r}}{\varepsilon_0} = -\frac{V_B}{d}$$

Surface $1r$ charge density

$$\sigma_{1r} = -\frac{\varepsilon_0 V_B}{d}$$

$$E_{2n} = \frac{\sigma_{2\ell}}{\varepsilon_0} = \frac{V_B}{d}$$

Surface 2ℓ charge density

$$\sigma_{2\ell} = +\frac{\varepsilon_0 V_B}{d}$$

Surface charge

$$Q_{2\ell} = -Q_{1r} = \sigma A = \frac{\varepsilon_0 V_B A}{d}$$

Force on each plate

$$F_2 = Q_2 (E_{2n})_{non\text{-}q_2} = \frac{1}{2}\frac{\sigma_2^2}{\varepsilon_0} A$$

Parallel plates (cont'd)

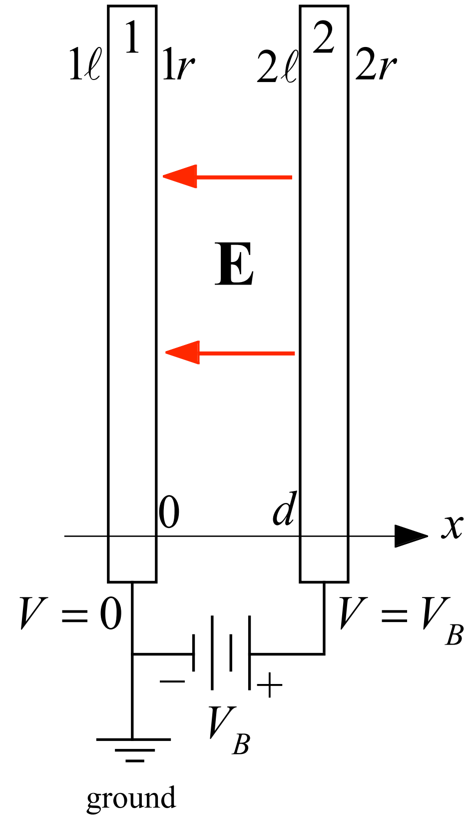
Capacitance

Two ways to know the electric field

$$E = \frac{\sigma}{\varepsilon_0}; \quad E = \frac{V_B}{d}$$

Capacitance is a geometrical factor relating the charge on a conductor and its potential.

$$Q = \sigma A = \frac{\varepsilon_0 A}{d} V_B = C V_B \qquad C = \frac{\varepsilon_0 A}{d}$$



Energy storage

$$U = \frac{\varepsilon_0}{2} \int E^2 d^3 x' = \frac{\varepsilon_0}{2} \frac{V_B^2}{d^2} A d = \frac{1}{2} \frac{\varepsilon_0 A}{d} V_B^2 \qquad U = \frac{1}{2} C V_B^2$$